Properties of Selick's filtration of the double suspension

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Outline

1. Selick's filtration $F_k(n)$
   1.1. Preliminaries about $\Omega^2 S^{2n+1}$
   1.2. Construction of $F_k(n)$
   1.3. Question related to $F_k(n)$
2. Fibrations related to $F_k(n)$
   2.1. Filtration of the classifying space BW_n
   2.2. Stable splitting coming from the fibrations
3. Global properties
   3.1. Homotopy exponents
   3.2. Our results
   3.3. Homotopy associativity and homotopy commutativity
   3.3. Further questions

- Working in the homotopy category of odd prime p-local spaces and maps.
- Serre's p-local decomposition

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- What $\Omega^2 S^{2n+1}$ is good for?

- For $\Omega S^{2n+1} = \Omega\Sigma S^{2n}$, $H_*(\Omega S^{2n+1}) = T(x_{2n})$ is torsion free.
- For $\Omega^2 S^{2n+1} = \Omega^2\Sigma^2 S^{2n-1}$, there is torsion in $H_*(\Omega^2 S^{2n+1})$.

- (1956, Toda) The first differential of the EHP spectral sequence in some metastable range

$$d_1 \colon \pi_r(\Omega^2 S^{2np+1}) \longrightarrow \pi_r(S^{2np-1})$$

satisfying that $\pi_r(S^{2np-1}) \xrightarrow{E_*^2} \pi_r(\Omega^2 S^{2np+1}) \xrightarrow{d_1} \pi_*(S^{2np-1})$ is multiplication by p.

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The differential d_1 depends on two fibre sequences:

- $\Omega^2 S^{2n+1} \xrightarrow{\vartheta} J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2n+1} \xrightarrow{H_p} \Omega S^{2n+1}$
- $S^{2n-1} \xrightarrow{i_1} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1}$

- (1979, Cohen-Moore-Neisendorfer) Constructed a map $\pi\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ such that

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\ \pi\ } & S^{2n-1} \\ & {\scriptstyle p}\searrow & \downarrow {\scriptstyle E^2} \\ & & \Omega^2 S^{2n+1} \end{array}$$

- This diagram was inductively used to determine the odd p-primary homotopy exponent of S^{2n+1}.

- Mod-p homology of $\Omega^2 S^{2n+1}$

$$H_*(\Omega^2 S^{2n+1};\mathbb{Z}/p) \cong (\otimes_{i=0}^{\infty} E(x_{2np^i-1})) \otimes (\otimes_{j=1}^{\infty} \mathbb{Z}/p[y_{2np^j-2}])$$

- Bocksteins: $\beta(x_{2np^i-1}) = y_{2np^i-2}$ for $i \geq 1$.
- Steenord operations: $\mathcal{P}^1_*(y_{2np^j-2}) = y^p_{2np^{j-1}-2}$ for $j \geq 2$.

Selick's filtration of $\Omega^2 S^{2n+1}$

- Homology filtration F_k of $H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$
  - $F_{2k-1} = (\otimes_{i=0}^{k-1} E(x_{2np^i-1})) \otimes (\otimes_{j=1}^{k} \mathbb{Z}/p[y_{2np^j-2}])$ for $k \geq 1$.
  - $F_{2k} = (\otimes_{i=0}^{k} E(x_{2np^i-1})) \otimes (\otimes_{j=1}^{k} \mathbb{Z}/p[y_{2np^j-2}])$ for $k \geq 0$.

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  - $F_{2k} = (\otimes_{i=0}^{k} E(x_{2np^i-1})) \otimes (\otimes_{j=1}^{k} \mathbb{Z}/p[y_{2np^j-2}])$ for $k \geq 0$.
- Geometric realization of F_k gives an H-filtration $F_k(n)$ of $\Omega^2 S^{2n+1}$

$$F_{-1}(n) \longrightarrow F_0(n) \longrightarrow \cdots \longrightarrow F_{k-1}(n) \longrightarrow F_k(n) \longrightarrow \Omega^2 S^{2n+1}$$

with $H_*(F_k(n); \mathbb{Z}/p) \cong F_k$.

- Let $F_{-1}(n) = \{*\}$ and $F_0(n) = S^{2n-1}$.
- Suppose that for $0 \leq i \leq k$ and any n, $F_i(n)$ has been constructed along with the H-maps

$$F_{-1}(n) \longrightarrow F_0(n) \longrightarrow \cdots \longrightarrow F_{i-1}(n) \longrightarrow F_i(n) \longrightarrow \Omega^2 S^{2n+1}.$$

Define $F_{k+1}(n)$ and the H-map $F_{k+1}(n) \longrightarrow \Omega^2 S^{2n+1}$ by the homotopy pullback

$$\begin{array}{ccc} F_{k+1}(n) & \longrightarrow & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \Omega H \\ F_{k-1}(np) & \longrightarrow & \Omega^2 S^{2np+1} \end{array}$$

where $H\colon \Omega S^{2n+1} \longrightarrow \Omega S^{2np+1}$ is the p^{th} James-Hopf invariant. By universality of homotopy pullback, the H-map $F_k(n) \longrightarrow F_{k+1}(n)$ can be constructed.

- (1956, Toda) $\qquad H_*(\Omega J_{p^k-1}(S^{2n}); \mathbb{Z}/p) \cong F_{2k-1}$
  where $J_{p^k-1}(S^{2n})$ is the $(p^k-1)^{th}$-filtration of the James construction $J(S^{2n}) \simeq \Omega\Sigma S^{2n}$.
- The calculation shows that $F_{2k-1}(n) \simeq \Omega J_{p^k-1}(S^{2n})$.

Three fibrations

- (1988, Gray) Gray's fibration

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$$

- (1956, Toda) Toda's fibration

$$S^{2n-1} \xrightarrow{i_1} \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1}$$

- (1981, Selick) Selick's fibration

$$S^{2n-1} \xrightarrow{i_2} F_2(n) \longrightarrow S^{2np-1}\{p\}$$

where $S^{2np-1}\{p\}$ is the fiber of the degree p map on S^{2np-1}.

- **Conjecture:** BW_n is a loop space. (related to Anick's space $T^{2np+1}(p)$)

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- **Question:** What connects Toda's, Selick's and Gray's fibrations? Is there any fibration of the type

$$S^{2n-1} \xrightarrow{i_k} F_k(n) \longrightarrow \square?$$

Study BW_n

- There is mod-p homology

$$H_*(BW_n;\mathbb{Z}/p) \cong (\otimes_{i=1}^{\infty} E(x_{2np^i-1})) \otimes (\otimes_{j=1}^{\infty} \mathbb{Z}/p[y_{2np^j-2}])$$

  with $\beta(x_{2np^i-1}) = y_{2np^i-2}$ for $i \geq 1$ and $\mathcal{P}^1_*(y_{2np^j-2}) = y^p_{2np^{j-1}-2}$ for $j \geq 2$.
- Homology filtration M_k of $H_*(BW_n;\mathbb{Z}/p)$
  - $M_{2k-1} = (\otimes_{i=1}^{k-1} E(x_{2np^i-1})) \otimes (\otimes_{j=1}^{k} \mathbb{Z}/p[y_{2np^j-2}])$ for $k \geq 1$.
  - $M_{2k} = (\otimes_{i=1}^{k} E(x_{2np^i-1})) \otimes (\otimes_{j=1}^{k} \mathbb{Z}/p[y_{2np^j-2}])$ for $k \geq 1$.

Theorem. (1) There is an H-filtration of BW_n

$$\{*\} = M_0(n) \longrightarrow M_1(n) \longrightarrow \cdots \longrightarrow M_{k-1}(n) \longrightarrow M_k(n) \longrightarrow BW_n$$

such that $H_*(M_k(n); \mathbb{Z}/p) \cong M_k$ for $k \geq 1$.

(2) There is a commutative diagram with rows being H-fibrations

$$\begin{array}{llcccc}
(k-1)^{th} \text{ fibration:} & S^{2n-1} & \xrightarrow{i_{k-1}} & F_{k-1}(n) & \xrightarrow{\nu_{k-1}} & M_{k-1}(n) \\
& \| & & \downarrow & & \downarrow \\
k^{th} \text{ fibration:} & S^{2n-1} & \xrightarrow{i_k} & F_k(n) & \xrightarrow{\nu_k} & M_k(n) \\
& \| & & \downarrow & & \downarrow \\
\infty^{th} \text{ fibration:} & S^{2n-1} & \xrightarrow{E^2} & \Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n.
\end{array}$$

- **Theorem.** There is a stable splitting

$$\Sigma^2 F_k(n) \simeq \Sigma^2(S^{2n-1} \times M_k(n)).$$

Sketch of proof.

(1) $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma X \wedge Y$.
(2) Define a map
$\beta\colon F_k(n) \xrightarrow{\triangle} F_k(n) \times F_k(n) \xrightarrow{1\times\nu_k} F_k(n) \times M_k(n)$

(3) The composite

$$\Sigma^2 F_k(n) \xrightarrow{\Sigma^2\beta} \Sigma^2(F_k(n)\times M_k(n)) \xrightarrow{\gamma} \Sigma^2(S^{2n-1}\times M_k(n))$$

is a homotopy equivalence, where the map γ is the composite

$$\Sigma^2(F_k(n)\times M_k(n)) \simeq \Sigma^2 F_k(n)\vee\Sigma^2 M_k(n)\vee\Sigma^2 F_k(n)\wedge M_k(n) \longrightarrow \Sigma^2\Omega^2 S^{2n+1}\vee\Sigma^2 M_k(n)\vee\Sigma^2\Omega^2 S^{2n+1}\wedge M_k(n) \xrightarrow{ev\vee 1\vee(ev\wedge 1)} S^{2n+1}\vee\Sigma^2 M_k(n)\vee S^{2n+1}\wedge M_k(n)\simeq\Sigma^2(S^{2n-1}\times M_k(n))$$

where $ev\colon \Sigma^2\Omega^2 S^{2n+1}\longrightarrow S^{2n+1}$ is the evaluation map.

- **Corollary.**

  (1) Recover Gray's (1988) result by letting $k \to \infty$

  $$\Sigma^2\Omega^2 S^{2n+1} \simeq \Sigma^2(S^{2n-1} \times BW_n)$$

  (2) $\Sigma^2\Omega J_{p-1}(S^{2n}) \simeq \bigvee_{i=0}^{\infty} S^{(2np-2)i+2n+1}$

  (3) $\Sigma^2 F_2(n) \simeq$
  $S^{2n+1} \vee (\bigvee_{i=1}^{\infty} P^{(2np-2)i+3}(p)) \vee (\bigvee_{i=1}^{\infty} P^{(2np-2)i+2n+2}(p))$
  where $P^m(p)$ is the mod p Moore space.

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- The classical Lie groups of low-rank by Theriault and some homogeneous spaces by Grbic and Z.

- Not too much is known about the homotopy exponents of many other spaces.

Theorem. Let $k \geq 1$. Then we have

- for $n \geq 1$, $p^{np^k - 2} \leq \exp(F_{2k-1}(n)) \leq p^{np^k}$;
- for $n \geq 2$, $p^{n-2} \leq \exp(F_{2k}(n)) \leq p^n$;
- for $n = 1$, $p \leq \exp(F_{2k}(n)) \leq p^2$.

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- for $n = 1$, $p \leq \exp(F_{2k}(n)) \leq p^2$.

The proof depends on two commutative diagrams:

$$\begin{array}{ccc} \Omega F_{2k}(n) & \xrightarrow{p^2} & \Omega F_{2k}(n) \\ \downarrow & & \| \\ \Omega S^{2n-1} & \xrightarrow{p\circ\Omega j_{2k}} & \Omega F_{2k}(n). \end{array}$$

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$$\begin{array}{ccc} \Omega^2 J_{p^k-1}(S^{2n}) & \xrightarrow{p^3} & \Omega^2 J_{p^k-1}(S^{2n}) \\ \downarrow & & \| \\ \Omega S^{2n-1} \times \Omega^2 S^{2np^k-1} & \xrightarrow{(p\circ\Omega j_{2k-1})\cdot(p^2\circ\Omega^2 c_k)} & \Omega J_{p^k-1}(S^{2n}). \end{array}$$

Consider the global properties of $F_k(n)$ and $M_k(n)$.

Definition 1. An H-space X with multiplication $\mu\colon X \times X \longrightarrow X$ is homotopy commutative if $\mu \simeq \mu \circ T$ where $T\colon X \times X \longrightarrow X \times X$ is the interchange map.

Definition 2. A space X is a homotopy associative H-space if there exist two maps $M_2\colon X \times X \longrightarrow X$ and $M_3\colon I \times X \times X \times X \longrightarrow X$ such that

1. $M_2(*, x) = x = M_2(x, *)$ for $x \in X$,
2. M_3 is a homotopy between $M_2 \circ (M_2 \times 1)$ and $M_2 \circ (1 \times M_2)$,
3. $M_3(t, *, x, y) = M_3(t, x, *, y) = M_3(t, x, y, *) = M_2(x, y)$ for $t \in I$ and $x, y \in X$.

Two known results

- (1989, Gray) $F_{2k-1}(n) \simeq \Omega J_{p^k-1}(S^{2n})$ is a homotopy associative, homotopy commutative H-space for $p \geq 3$.
- (2006, Grbić) $F_2(n)$ is a homotopy associative, homotopy commutative H-space for $p > 3$.

Question: Including $F_2(n)$, is $F_{2k}(n)$ $(k \geq 0)$ a homotopy associative, homotopy commutative H-space?

Theorem. $F_{2k}(n)$ $(k \geq 0)$ is a homotopy associative H-space for $p > 3$.

Sketch of proof.

(1) Show that $E^2\colon S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ is a homotopy associative map for $p > 3$.

(2) Zabrodsky's lemma: the pullback of two homotopy associative maps is a homotopy associative H-space.

(3) Consider the homotopy pullback

$$\begin{array}{ccc} F_{2k}(n) & \longrightarrow & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow (\Omega H)^{\circ k} \\ S^{2np^k-1} & \xrightarrow{E^2} & \Omega^2 S^{2np^k+1}. \end{array}$$

Note. $F_{2k}(n)$ $(k \geq 0)$ actually admits an A_{p-1}-structure.

Questions

Q1. Gray has shown that $F_{2k-1}(n)$ is a homotopy associative, homotopy commutative H-space. Whether $F_{2k}(n)$ is also a homotopy associative, homotopy commutative H-space or not?

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Note. The associativity has already been shown by us.

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Note. The associativity has already been shown by us.

Q2. It is already known that $F_k(n)$ is atomic for $n > 1$. Is $\Omega F_k(n)$ or $\Omega^2 F_k(n)$ still atomic for $n > 1$?

THANK YOU