Arbitrarily small spectral gaps for random hyperbolic surfaces with many cusps

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joint with Yunhui Wu

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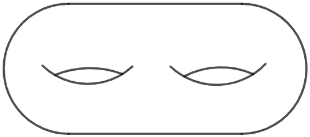
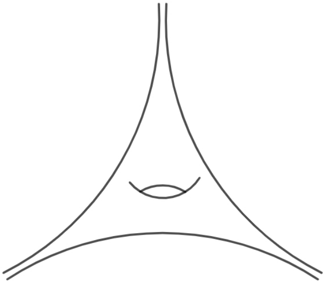


$S_{2,0}$



$S_{1,3}$

Assume X is a complete hyperbolic surface with finite area, the Cheeger constant $h(X)$ of X is defined as

$$h(X) = \inf_{\alpha} \frac{\ell(\alpha)}{\min\{\mathrm{Area}(A), \mathrm{Area}(B)\}},$$

where α runs over all curves such that $X \setminus \alpha = A \cup B$.

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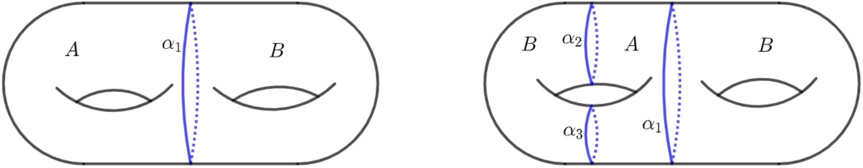

$\alpha = \alpha_1$ $\qquad$ $\alpha = \bigcup_{i=1}^{3} \alpha_i$

Assume X is a compact hyperbolic surface.

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As a direct corollary

$$\limsup_{g \to \infty} \lambda_1(X_g) \leq \frac{1}{4}.$$

The first eigenvalue is related to the Cheeger constant:

Theorem (Cheeger-Buser)

Assume X is a compact hyperbolic surface, then

$$\frac{1}{4}h(X)^2 \leq \lambda_1(X) \leq 2h(X) + 10h(X)^2.$$

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The following fundamental question remains open:

Question

Does a complete non-compact hyperbolic surface of finite area always have a non-zero eigenvalue?

Assume X is a complete non-compact hyperbolic surface, instead of λ_1, consider Rayleigh quotient $\mathrm{RayQ}(X)$:

$$\mathrm{RayQ}(X) = \inf_{f \in L^2(X), \int_X f = 0} \frac{\int_X |\nabla f|^2}{\int_X f^2}.$$

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Similar to the case of compact

Theorem (Buser 82')

Assume X is a complete non-compact hyperbolic surface with finite area, there exists a universal constant c such that

$$\frac{1}{4} h(X)^2 \leq RayQ(X) \leq c \cdot h(X).$$

Theorem

Assume X is a complete non-compact hyperbolic surface with finite area. If

$$RayQ(X) < \frac{1}{4},$$

then X has a non-zero first eigenvalue $\lambda_1(X)$ with

$$\lambda_1(X) = RayQ(X).$$

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small Cheeger constant $\Longrightarrow$ small Rayleigh quotient
$\Longrightarrow$ the first eigenvalue exists and small.

Consider

$$\Gamma(N) \overset{def}{=} \{A \in \mathrm{SL}(2,\mathbb{Z}); A \equiv I_2 (\mathrm{mod}\ N)\}.$$

Then $X(N) = \mathbb{H}/\Gamma(N)$ is a non-compact hyperbolic surface with genus $g(N)$ and $n(N)$ cusps, where

$$g(N) = 1 + \frac{N^3}{24}\left(1 - \frac{6}{N}\right)\prod_{p|N}\left(1 - \frac{1}{p^2}\right) \asymp N^3;$$

$$n(N) = \frac{N^2}{2}\prod_{p|N}\left(1 - \frac{1}{p^2}\right) \asymp N^2.$$

Selberg 65' proved that

$$\lambda_1(X(N)) \geq \frac{3}{16}.$$

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Kim-Sanark 03': $\frac{975}{4096}$

Question

Is there a sequence $\{X_g\}_{g\geq 2}$ *of compact hyperbolic surfaces with genus* g*, such that*

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It was solved by Hide-Magee 21'.

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Hence Weil-Petersson metric induces a probability measure on $\mathcal{M}_{g,n}$.

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1. Guth-Parlier-Young 11': Bers constant;
2. Mirzakhani 13',Wu-Xue 22': diameter;
3. Mirzakhani-Petri 19: systole;
4. Mirzakhani 13',Palier-Wu-Xue 22', Nie-Wu-Xue 23': separating systole;
5. Monk 22': Weyl law;
6. Wu-Xue 22': prime geodesic theorem;
7. Rudnick 23': GOE;
8. He-S.-Wu-Xue 23': non-simple systole,

...

Conjecture

For any $\epsilon > 0$,

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{wp}}^{g}\left(X \in \mathcal{M}_g;\ \lambda_1(X) > \frac{1}{4} - \epsilon\right) = 1.$$

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3. Anantharaman-Monk 23': $\frac{2}{9}$.

Theorem (Hide 22')

Assume $n(g) = O(g^{\alpha})\left(0 \le \alpha < \frac{1}{2}\right)$. *Then*

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{wp}}^{g,n(g)}\left(X \in \mathcal{M}_{g,n(g)};\ Spec(\Delta_X) \cap (0, c(\alpha)-\epsilon) = \emptyset\right) = 1,$$

where $c(\alpha) = \frac{1}{4} - \left(\frac{2\alpha+1}{4}\right)^2$. $c(0) = \frac{3}{16}$, $c\left(\frac{1}{2}\right) = 0$.

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Based on the examples of $X(N)$ ($n \asymp g^{2/3}$),

Question

If $n(g)$ *grows significiantly faster than* $\sqrt{g}$, *asymptotically does a generic surface in* $\mathcal{M}_{g,n(g)}$ *have a uniform positive spectral gap as* $g \to \infty$.

Theorem (S.-Wu 23')

If $n(g)$ satisfies that

$$\lim_{g\to\infty}\frac{n(g)}{\sqrt{g}}=\infty \text{ and } \lim_{g\to\infty}\frac{n(g)}{g}=0,$$

then for any $\epsilon>0$

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2. $\{X(N)\}_{N\geq 3}$ are exceptional in the moduli space.

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Theorem (S.-Wu 23')

Assume $n(g)$ satisfies that

$$\lim_{g\to\infty} \frac{n(g)}{\sqrt{g}} = a \in (0,\infty).$$

Then for any $0 < C < \frac{\log 2}{\sqrt{4\pi(\log 2+\pi)}}$,

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}\left(X \in \mathcal{M}_{g,n(g)};\, h(X) \leq \frac{C}{\sqrt{1+C^2}}\right) = 1 - e^{-\lambda(a,C)},$$

where $\lambda(a,C) = \frac{a^2}{4\pi^2}(\cosh \pi C - 1)$.

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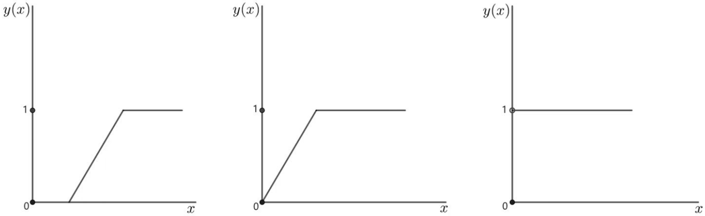
3. For any $x > 0$, denote by

$$y(x) = \liminf_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g,n(g)}\left(X \in \mathcal{M}_{g,n(g)};\ \lambda_1(X) \le x\right).$$



$\lim_{g\to\infty} \frac{n(g)}{\sqrt{g}} = 0$ $\quad$ $\lim_{g\to\infty} \frac{n(g)}{\sqrt{g}} = a$ $\quad$ $\lim_{g\to\infty} \frac{n(g)}{\sqrt{g}} = \infty$ and $\lim_{g\to\infty} \frac{n(g)}{g} = 0$

According to the result of Zograf 84':

If $n(g)$ satisfies that $\lim_{g\to\infty} \frac{n(g)}{g} = \infty$, then for any $\epsilon > 0$

$$\limsup_{g\to\infty} \sup_{X\in\mathcal{M}_{g,n(g)}} \lambda_1(X) < \epsilon.$$

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The proof is based on the method of Yang-Yau 80'.

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Study the asymptotic behavior of Weil-Petersson volume $V_{g,n(g)}$, which is related to the theory of intersection numbers.

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Proposition (S.-Wu 23')

$$\limsup_{g+n\to\infty} \frac{V_{g,n}^2}{V_{g,n-1}V_{g,n+1}} \leq 1.$$

For any surface $X \in \mathcal{M}_{g,n}$, denote by $\mathcal{N}_{0,3}(X, L)$ the set consisting of all simple closed geodesics in X such that

$$X \setminus \alpha \simeq S_{0,3} \cup S_{g,n-1} \text{ and } \ell(\alpha) \leq L.$$

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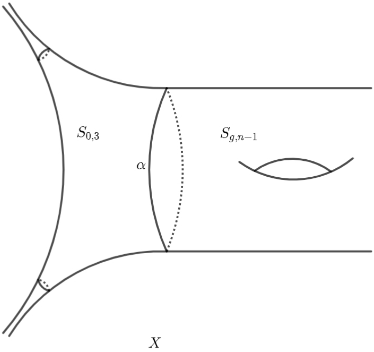

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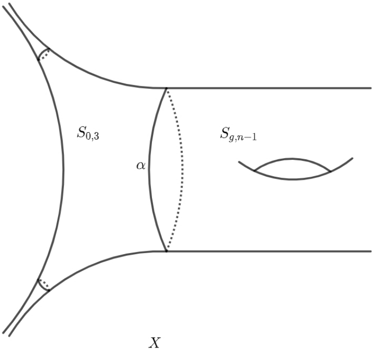
Denote by $N_{0,3}(X, L) = |\mathcal{N}_{0,3}(X, L)|$, then

$$N_{0,3}(\cdot, L) : \mathcal{M}_{g,n} \to \mathbb{Z}_{\geq 0}$$

is a random variable on the probability space $\mathcal{M}_{g,n}$.



Set $L(g) = \left(\frac{\sqrt{g}}{n(g)}\right)^{\frac{1}{2}}$. By direct calculation,

$$\mathbb{E}_{g,n(g)}[N_{0,3}(\cdot, L(g))] \asymp \frac{\sqrt{g} \cdot n(g)}{g + n(g)}.$$

1. If $\lim_{g\to\infty} \frac{n(g)}{\sqrt{g}} = \infty$. Then as $g \to \infty$,

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It follows that as $g \to \infty$, a generic surface $X \in \mathcal{M}_{g,n(g)}$ has a simple closed geodesic α such that

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Hence $h(X)$ is very small $\Longrightarrow$ X has small non-zero eigenvalue.

2. If $\lim_{g\to\infty} \frac{n(g)}{\sqrt{g}} = 0$, then as $g \to \infty$,

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The behavior is totally different. In this case, we complete proof by following Mirzakhani's method.

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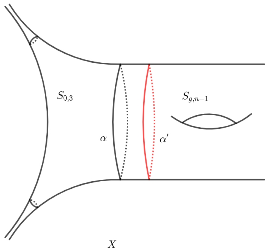
One may prove that for small $L > 0$, random variable $N_{0,3}(\cdot, L)$ converges to a Poisson distribution.

It follows that

$$\lim_{g\to\infty} \text{Prob}_{\text{WP}}^{g,n(g)} \left(X \in \mathcal{M}_{g,n(g)};\ N_{0,3}(X,L) \geq 1 \right) = 1 - e^{-\lambda(a,2\pi L)}.$$

In this case, we may prove that the Cheeger constant is realized by some simple closed curve α' "near" α. Then the proof is complete by some hyperbolic calculation.



Yang Shen, Yunhui Wu: **Arbitrarily small spectral gaps for random hyperbolic surfaces with many cusps**, *arXiv:2203.15681, 2023*.

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Thanks for your listening!