Toric spaces and face enumeration on simplicial manifolds

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- Motivation: the g-conjecture for simplicial spheres and the generalized g-conjecture for simplicial manifolds in the theory of face enumeration.

- Goal: to study these enumeration problems from the point of view of toric topology, and to give topological expositions for many fundmental results in algebraic combinatorics.

How the g-conjecture came about

This conjecture comes from the following naive question:

Question

What are the possible face numbers of triangulations of spheres?

- There is only one zero-dimensional sphere and it consists of two disjoint points.
- The triangulations of 1-spheres are boundaries of polygons, having n vertices and n edges for $n \geq 3$.
- The 2-spheres with n vertices have $3n-6$ edges and $2n-4$ faces for any $n \geq 4$. This follows from Euler's formula.

Euler's fomula

Let V, E, F are the numbers of vertices, edges and faces of a polytope respectively. Then

$$V - E + F = 2.$$

f-vectors

Let K be a simplicial complex of dimension $d-1$. Denote by $f_i(K)$ the number of i-dimensional faces of K. The integer sequence $(f_0, f_1, \ldots, f_{d-1})$ is known as the f-vector of K.

Euler-Poincaré formula

$f_0 - f_1 + f_2 + \cdots + (-1)^{d-1} f_{d-1} = e(K)$ is a topological invariant called the Euler number of K

For spheres, we have

$$e(S^{d-1}) = 1 - (-1)^d.$$

For simplicial 2-spheres, there is also a linear relation:

$$2f_1 = 3f_2.$$

h-vectors & Dehn-Sommerville relations

Question

What are the relations between the face numbers of simplicial spheres in higher dimensions?

The h-vector of a $d-1$-dimensional simplicial complex K is the integer vector $(h_0, h_1, \ldots, h_d)$ defined from the equation

$$h_0 t^d + \cdots + h_{d-1} t + h_d = (t-1)^d + f_0 (t-1)^{d-1} + \cdots + f_{d-1}.$$

Here is a trick, like Pascal's triangle (using subtractions instead of additions), can be used to compute the h-vector.

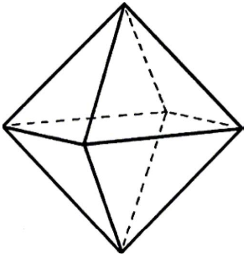

Figure: Octahedron with $f = (6, 12, 8)$

		1	
	1	6	
1	5	12	
1	4	7	8

$h = (1 \quad 3 \quad 3 \quad 1)$

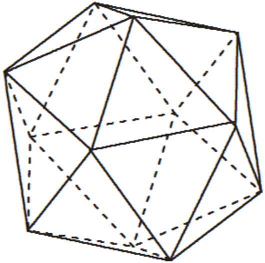

Figure: Icosahedron with $f = (12, 30, 20)$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 12 & & & \\
 & & 1 & & 11 & & 30 & & \\
 & 1 & & 10 & & 19 & & 20 & \\
h = (1 & & 9 & & 9 & & 1) & &
\end{array}
$$

Dehn-Sommerville relations

If K is a triangulation of S^{d-1}, then $h_i = h_{d-i}$ for $0 \le i \le d$.

- These equations were found by Dehn and Sommerville for simplicial polytopes, and were extended to simplicial spheres, even to simplicial homology spheres by Klee in 1964.
- The equation $1 = h_0 = h_d$ is equivalent to the Euler-Poincaré formula

$$f_0 - f_1 + f_2 + \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d.$$

- (Klee) The Dehn-Sommerville relations are the most general linear equations satisfied by the face numbers of all simplicial spheres.
- Since for triangulations of S^{d-1} the number of Dehn-Sommerville equations is $[\frac{d+1}{2}]$, the number of vertices doesn't determine all the face numbers for simplicial spheres of $\dim \ge 3$.

Question

Given any sequence $(h_0 = 1, h_1, \ldots, h_d)$ of positive integers such that $h_i = h_{d-i}$, is there always a simplicial sphere having this sequence as its h-vector?

- The answer is 'no', because there are some other inequality relations between the h-numbers of triangulated spheres.

The Upper Bound Conjecture

Upper Bound Conjecture

For any triangulated $(d-1)$-dimensional sphere K with m vertices, the h-vector satisfies the inequalities

$$h_i \leq \binom{m-d+i-1}{i}.$$

- In 1975, Stanley proved the UBC by showing that h-vectors of triangulated spheres are M-sequences.

LBC & GLBC

Lower Bound Conjecture

Let K be a simplicial $(d-1)$-sphere of with $d \geq 3$. Then $f_1 \geq df_0 - \binom{d+1}{2}$, or in other words, $h_2 \geq h_1$.

- LBC was proved by Barnette in 1970 for simplicial polytopes, and extended to the most general case of homology manifolds by Kalai in 1987.

Generalized Lower Bound Conjecture

Let K be a simplicial $(d-1)$-sphere with $d \geq 6$. Then

$$h_0 \leq h_1 \leq h_2 \leq \cdots \leq h_{[d/2]}.$$

- GLBC is implied by the g-conjecture.

Macaulay condition

For any two positive integers a and i there is a unique way to write

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. Define the ith pseudopower of a as

$$a^{\langle i \rangle} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j+1}.$$

A sequence of nonnegative integers $(a_0, a_1, a_2, \ldots)$ satisfies $a_0 = 1$ and $a_{i+1} \leq a_i^{\langle i \rangle}$ for $i \geq 1$ is called an M-sequence.

Theorem (Macaulay, 1927)

A sequence $(a_0, a_1, a_2 \ldots)$ of nonnegative integers is an M-sequence if and only if there exists a connected commutative graded $\mathbf{k}$-algebra R over a field $\mathbf{k}$ such that R is generated by degree-one elements and $\dim_{\mathbf{k}} R_i = a_i$ for all i.

The g-conjecture

g-vector

For a simplicial compelx K of dimension $d-1$, the *g-vector* $(g_0, g_1, \ldots, g_{[d/2]})$ of K is defined by $g_0 = h_0$ and $g_i = h_i - h_{i-1}$ for $i = 1, \ldots, [d/2]$.

McMullen's g-conjecture (1971)

An integer vector $(h_0, h_1, \ldots, h_d)$ is the h-vector of the boundary sphere of a simplicial d-polytope if and only if

(i) $h_i = h_{d-i}$ for $i = 0, 1, \ldots, d$;

(ii) the g-vector is a M-sequence.

- Less than ten years later McMullen's g-conjecture was proved. Now it is known as the g-Theorem.

- Billera and Lee (1981) proved the sufficiency of McMullen's conditions. They established that for every M-vector there was a simplicial polytope with the given g-vector.
- Stanley (1980) gave a topological proof of the necessity of McMullen's conditions. His proof used deep results from algebraic geometry, especially the theory of toric varieties.
- In 1993, McMullen gave another more elementary but complicated proof of the necessity part of g-theorem.
- The idea behind both Stanley's and McMullen's proofs was to find an algebra whose Hilbert function equals the g-vector of the polytope.

Face rings and its artinian reduction

Let K be a simplicial complex on $[m]$. The face ring or Stanley-Reisner ring of K over a field $\mathbf{k}$ is the quotient ring

$$\mathbf{k}[K] := \mathbf{k}[x_1, \ldots, x_m]/I_K.$$

Here $I_K := (x_{i_1}x_{i_2}\cdots x_{i_k} : \{i_1, i_2, \ldots, i_k\} \notin K)$.

A set $\Theta = \{\theta_1, \ldots, \theta_d\}$ of $d = \dim K + 1$ linear forms in $\mathbf{k}[K]$ is called an linear system of parameters, or lsop, for $\mathbf{k}[K]$, if $\dim_{\mathbf{k}}(\mathbf{k}[K]/\Theta) < \infty$. In this case, $\mathbf{k}[K]/\Theta$ is also called an artinian reduction of $\mathbf{k}[K]$.

Cohen-Macaulay complexes

The face ring $\mathbf{k}[K]$ is a Cohen-Macaulay ring if for any lsop Θ, $\mathbf{k}[\Delta]/\Theta$ is a free $\mathbf{k}[\theta_1, \cdots, \theta_d]$ module. In this case, K is called a Cohen-Macaulay complex over $\mathbf{k}$.

Theorem (Stanley, 1975)

Let K be a $(d-1)$-dimensional Cohen-Macaulay complex and let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be an lsop for $\mathbf{k}[K]$. Then

$$\dim_{\mathbf{k}}(\mathbf{k}[K]/\Theta)_i = h_i(K), \quad \text{for all } 0 \leq i \leq d.$$

Theorem (Reisner, 1976)

K is Cohen-Macaulay (over $\mathbf{k}$) if and only if for every face $\sigma \in K$ (including $\sigma = \varnothing$) and $i < \dim \mathrm{lk}_\sigma K$, we have $\widetilde{H}_i(\mathrm{lk}_\sigma K; \mathbf{k}) = 0$.

- Simplicial spheres are Cohen-Macaulay.

Toric varieties

Given a rational simplicial fan $\Sigma \subset \mathbb{R}^d$. Each ray of Σ is generated by a primitive vector $\boldsymbol{\lambda}_i = (\lambda_{1i}, \dots, \lambda_{di}) \in \mathbb{Z}^d$. Let K_Σ to be the underlying simplicial complex of Σ. Then the vectors $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m$ define an lsop for $\mathbb{Q}[K_\Sigma]$:

$$\Theta = \{\theta_i = \lambda_{i1}x_1 + \cdots + \lambda_{im}x_m\}_{i=1}^d.$$

Theorem (Danilov, 1978)

Let Σ be a rational complete simplicial fan, and let X_Σ be the corresponding toric variety. Then there is a ring isomorphism

$$H^*(X_\Sigma;\mathbb{Q}) \cong \mathbb{Q}[K_\Sigma]/\Theta, \quad H^{2i}(X_\Sigma;\mathbb{Q}) \cong (\mathbb{Q}[K_\Sigma]/\Theta)_i.$$

In particular, if Σ is regular, then the above isomorphism also holds for integeral cohomology.

Hard Lefschetz Theorem

Hard Lefschetz Theorem

Let (M, ω) be a compact Kähler manifold of complex dimensinon n with a Kähler form $\omega \in H^2(M; \mathbb{C})$. For each $k \leq n$ the multiplication map

$$\cdot\omega^k : H^{n-k}(M; \mathbb{C}) \to H^{n+k}(M; \mathbb{C})$$

is an isomorphism.

- The Hard Lefschetz Theorem also holds for Kähler orbifolds by using intersection cohomology theory.
- Projective manifolds or orbifolds $M \subset \mathbb{P}^n$ are Kähler.
- If Σ is a simplicial fan corresponding to a simplicial polytope, then the toric variety X_Σ is a projective orbifold. Consequently the algebra $H^*(X_\Sigma)/\langle\omega\rangle$ will have $g(K_\Sigma)$ as its Hilbert function.

SLP & Algebraic g-conjecture

We say a simplicial $(d-1)$-sphere K has the strong Lefschetz property if there exists an lsop Θ for $\mathbf{k}[K]$ and a linear form ω such that the multiplication map

$$\cdot\omega^{d-2i} : (\mathbf{k}[K]/\Theta)_i \to (\mathbf{k}[K]/\Theta)_{d-i}$$

is an isomorphism for all $i \le d/2$.

Algebraic g-conjecture

Every simplicial sphere has the strong Lefschetz property.

- In 2018, Karim Adiprasito announced a proof of the algebraic g-conjecture in the paper:
  Combinatorial Lefschetz theorems beyond positivity, arXiv:1812.10454.

Weak Lefschetz property

In fact, to prove the g-conjecture, it is enough to prove the following weaker property holds.

We say a simplicial $(d-1)$-sphere K has the weak Lefschetz property if there is an lsop Θ and a linear form ω such that the multiplication map $\cdot\omega : (\mathbf{k}[K]/\Theta)_i \to (\mathbf{k}[K]/\Theta)_{i+1}$ has maximal rank, i.e. is injective or surjective, for all i.

Simple polyhedral complexes

Given a pure simplicial complex K on $[m]$, there is a dual simple polyhedral complex P_K. As a polyhedron, P_K is the cone over the barycentric subdivision K' of K.

- For each i-face $\sigma \in K$, let F_σ be the geometric realization of the poset
$$\{\tau \in K : \tau \geq \sigma\}.$$
Then F_σ is a face of P_K of codimension $i+1$. In particular, if v is a vertex of K, $F_v = \mathrm{st}_v K'$ is a facet of P_K.
- For each point $x \in P_K$, let $F(x)$ be the unique face of P_K which contains x in its relative interior.

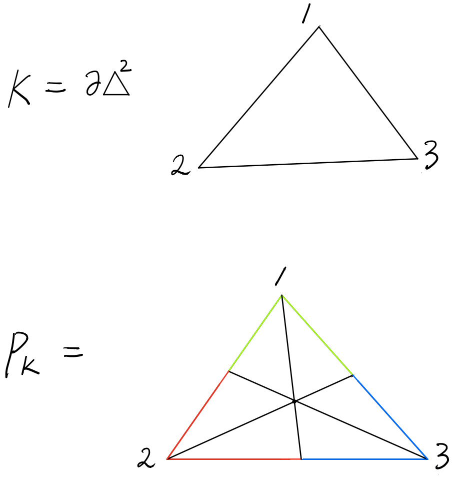

Figure: F_1, F_2, F_3 are the three facets of P_K

Toric spaces by D-J construction

Suppose $\dim K = d - 1$ and the vertex set of K is $[m]$. A map

$$\mathit{\Lambda} : [m] \to \mathbb{Z}^d,\ i \mapsto \boldsymbol{\lambda_i} = (\lambda_{1i}, \ldots, \lambda_{di})^T$$

is called a characteristic function if $\Theta = \{\theta_i = \lambda_{i1}x_1 + \cdots + \lambda_{im}x_m\}_{i=1}^d$ is an lsop for $\mathbb{Q}[K]$.

For the pair $(K, \mathit{\Lambda})$, there is a toric space defined by

$$M(K, \mathit{\Lambda}) = P_K \times T^d / \sim.$$

$(x, g) \sim (x', g')$ if and only if $x = x'$ and $g^{-1}g' \in G_{F(x)}$. If $F(x) = F_\sigma$ for some $\sigma = \{i_1, i_2, \ldots, i_k\}$, $G_{F(x)} \subset T^d$ is the subtorus corresponding to the sublattice spanded by $\{\boldsymbol{\lambda_{i_1}}, \boldsymbol{\lambda_{i_2}}, \ldots, \boldsymbol{\lambda_{i_k}}\}$.

Cohomology of $M(K, \Lambda)$

Theorem (Davis-Januszkiewicz, 1991)

If K is Cohen-Macaulay over $\mathbb{Z}$ and Λ gives a integral lsop Θ for $\mathbb{Z}[K]$ (i.e. the characteristic function is regular), then

$$H^*(M(K, \Lambda); \mathbb{Z}) \cong \mathbb{Z}[K]/\Theta$$

Theorem (F., 2020)

If K is Cohen-Macaulay over $\mathbb{Q}$ and Λ gives an lsop Θ for $\mathbb{Q}[K]$, then we have a ring isomorphism

$$H^*(M(K, \Lambda); \mathbb{Q}) \cong \mathbb{Q}[K]/\Theta$$

Moment-angel complexes

Let K be a simplicial complex on $[m]$. For each face $\sigma \in K$, let

$$D(\sigma) = \prod_{i=1}^{m} Y_i, \quad \text{where } Y_i = \begin{cases} D^2 \text{ if } & i \in \sigma, \\ S^1 \text{ if } & i \notin \sigma. \end{cases}$$

The space $\mathcal{Z}_K = \bigcup_{\sigma \in K} D(\sigma)$ is known as the moment-angle complex corresponding to K.

- $\mathcal{Z}_K$ is a topological manifold (resp. $\mathbf{k}$-homology manifod) if and only if K is a $\mathbb{Z}$-homology sphere (resp. $\mathbf{k}$-homology sphere).

A simplicial complex K is a $\mathbf{k}$-homology d-manifold if

$$H_*(\mathrm{lk}_\sigma K; \mathbf{k}) = H_*(S^{d-|\sigma|}; \mathbf{k}) \quad \text{for all } \varnothing \neq \sigma \in K.$$

K is a $\mathbf{k}$-homology sphere if it is a $\mathbf{k}$-homology manifold with the same $\mathbf{k}$-homology as S^d.

- Usually, the terminology "homology sphere" means a manifold having the homology of a sphere. Here we take it in the most relaxed sense.

Quotient constructions of $M(K, \Lambda)$

The characteristic function defines a map of lattices: $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^d$, which can be extended to a exact sequence

$$0 \to \mathbb{Z}^{m-d} \to \mathbb{Z}^m \xrightarrow{\Lambda} \mathbb{Z}^d \to G \to 0,$$

such that G is a finite group.

It can be shown that Λ induces an epimorphism of tori $\exp \Lambda : T^m \to T^d$ with kernel $\mathcal{K}_\Lambda = T^{m-d} \times G$.

The group $\mathcal{K}_\Lambda = T^{m-d} \times G$ acts almost freely and properly on $\mathcal{Z}_K$, and there is a T^d-equaviriant homeomorphism

$$\mathcal{Z}_K / \mathcal{K}_\Lambda \cong M(K, \Lambda).$$

If K is a rational homology $(d-1)$-sphere, M_K is a closed, orientable, rational homology $2d$-manifold, called a rational toric manifold.

Toric submanifolds of M_K

Let $\sigma \in K$ be a $(k-1)$-face, and let $\mathcal{S}_\sigma$ be the vertex set of $\mathrm{st}_\sigma K$. Then the T^d-subspace of M_K:

$$M_\sigma = (\mathcal{Z}_{\mathrm{st}_\sigma K} \times T^{m-|\mathcal{S}_\sigma|})/\mathcal{K}_\Lambda \subset M_K$$

is an orientable, rational D^{2k}-fibration, and the zero section N_σ is a rational toric $(2d-2k)$-manifold.

- Especially, if $i \in [m]$ is a vertex, then N_i is a divisor in the case of toric varieties.

Poincaré duality

Lemma (Poincaré duality)

If K is a rational homology $(d-1)$-sphere, then the map defined by

$$H^{2j}(M_K;\mathbb{Q}) \xrightarrow{[M_K]\frown} H_{2d-2j}(M_K;\mathbb{Q})$$

is an isomorphism for all $j \leq d$. Moreover, for any $(k-1)$-face $\sigma = \{i_1, \ldots, i_k\} \in K$, we have

$$[M_K] \frown \mathbf{x}_\sigma = [N_\sigma], \quad \text{where } \mathbf{x}_\sigma = x_{i_1} \cdots x_{i_k}.$$

$[N_\sigma] \in H_{2d-2k}(N_\sigma;\mathbb{Q})$ is the fundamental class of N_σ.

Corollary

$\mathbb{Q}[K]/\Theta$ is a Poincaré duality algebra.

Injection principles

Theorem (F., 2020)

Let K be a rational homology $(d-1)$-sphere. If $\mathbf{x}_{\sigma_1}, \ldots, \mathbf{x}_{\sigma_s}$ generates $(\mathbb{Q}[K]/\Theta)_k$, then for each $i \leq d-k$ we have an injection

$$(\mathbb{Q}[K]/\Theta)_i \to \bigoplus_{j=1}^{s} (\mathbb{Q}[\mathrm{st}_{\sigma_j} K]/\Theta)_i.$$

Moreover, if there is $\mathcal{S} \subset [m]$ such that $\sigma_j \cap \mathcal{S} \neq \varnothing$ for $1 \leq j \leq s$, then for each $i \leq d-k$ we also have an injection

$$(\mathbb{Q}[K]/\Theta)_i \to \bigoplus_{v \in \mathcal{S}} (\mathbb{Q}[\mathrm{st}_v K]/\Theta)_i.$$

From this we can derive a result of Swartz:

Theorem (Swartz, 2009)

If $\mathrm{lk}_v K$ has WLP for at least $m-d$ of the vertices v of K, then K satisfies the g-conjecture.

Toric spaces over simplicial manifolds

If K is a rational homology $(d-1)$-manifold, then $M_K - T^d$ is a rational open $2d$-manifold. Here T^d is the orbit of the coning point in P_K ($|P_K| = |CK|$). This can be seen from a decomposition of M_K:

$$M_K = (CK \times T^d) \cup (I \times K \times T^d / \sim).$$

Here I is the unit interval.

Theorem (F-, 2020)

The cohomology ring of M_K over a rational homology manifold K is

$$H^*(M_K;\mathbb{Q}) \cong \mathcal{R} \oplus \mathbb{Q}[K]/\Theta, \quad \mathcal{R}^k = \bigoplus_{q>0,\, 2p+q=k} \tbinom{d}{p+q} \widetilde{H}^{p-1}(K;\mathbb{Q}),$$

where $\mathcal{R}$ has trivial multiplication structure.

$$\text{Let } j^* : H^*(M_K, I \times K \times T^d/\sim) \to H^*(M_K), \quad \mathcal{J} = \bigoplus_{k=1}^{d-1} (\operatorname{Im} j^*)_{2k}$$

If K is an orientable rational homology manifold, then the quotient algebra $\mathcal{A} = H^*(M_K)/\mathcal{J}$ is a Poincaré duality algebra.

$$\dim \mathcal{A}^{2k} = h_k(K) - \binom{d}{k} \sum_{i=0}^{k-1} (-1)^i \widetilde{\beta}_{k-i-1}(K).$$

In fields of $\mathrm{char} = 0$, the above theorem is a topological interpretation of the following two important algebraic results:

Theorem (Schenzel, 1981)

Let K be a $\mathbf{k}$-homology $(d-1)$-manifold. Then,

$$h'(K) := \dim_{\mathbf{k}}(\mathbf{k}[K]/\Theta)_j = h_j(K) - \binom{d}{j}\sum_{i=1}^{j-1}(-1)^i \widetilde{\beta}_{j-i-1}(K;\mathbf{k}).$$

For a graded ring R, the socle of R is

$$\mathrm{Soc}(R) := \{x \in R : x \cdot R_+ = 0\}.$$

Theorem (Novik-Swartz, 2009)

Let K be a orientable $\mathbf{k}$-homology $(d-1)$-manifold, then for any lsop Θ the quotient algebra $\mathbf{k}[K]/(\Theta + J)$ is a Poincaré duality $\mathbf{k}$-algebra, where

$$J = \mathrm{Soc}(\mathbf{k}[K]/\Theta)_{<d}, \quad \dim J_i = \binom{d}{j}\widetilde{\beta}_{i-1}(K;\mathbf{k}).$$

Manifold g-conjecture

To generalize the g-conjecture for spheres to manifolds, Kalai introduced the h''-vectors. Let K be a $(d-1)$-dimensional simplicial complex, and let $h'_i(K) = \dim_{\mathbf{k}}(\mathbf{k}[K]/\Theta)_i$. Then

$$h''_i(K) = \begin{cases} h'_i(K) - \binom{d}{i}\widetilde{\beta}_{i-1}(K;\mathbf{k}) & \text{if } 0 \le i < d; \\ h'_d(K) & \text{if } i = d. \end{cases}$$

Kalai's manifold g-conjecture

If K is an orientable $(d-1)$-dimensional $\mathbf{k}$-homology manifold, then the vector

$$(g''_i := h''_i - h''_{i-1})_{i=0}^{[d/2]}$$

is an M-vector.

- If every $\mathbf{k}$-homology sphere has WLP, then Kalai's manifold g-conjecture holds (Novik-Swartz).

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Thanks!