Topological classification of complex vector bundles over 8-dimensional spin^c manifolds

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Motivation

- It is a classical topic in geometry and topology to classify vector bundles over manifolds.
- To classify holomorphic vector bundles (Schneider, "Holomorphic vector bundles on P_n",LNM770).
- To classify complex structures on real vector bundles.

Motivation

Denote by

- M: a closed oriented smooth $2n$-manifold.
- $\mathrm{Vect}^r_{\mathbb{C}}(M)$: the isomorphism classes of rank r smooth complex vector bundles over M.
- $H^*(M)$ the integral cohomology ring of M.

- If $r \geq n$, there is a natural bijection between $\mathrm{Vect}^r_{\mathbb{C}}(M)$ and $\mathrm{Vect}^n_{\mathbb{C}}(M)$ for any $r \geq n$.
- If $r = 1$, we have a bijection between $\mathrm{Vect}^1_{\mathbb{C}}(M)$ and $H^2(M)$ for any $n \geq 0$.

Motivation

M

- $n = 1$: $\mathrm{Vect}^1_{\mathbb{C}}(M)$
- $n = 2$: for $r = n = 2$, a classical result of Wu asserts that there is a bijection between $\mathrm{Vect}^2_{\mathbb{C}}(M)$ and $H^2(M) \times H^4(M)$. 1952
- $n = 3$: Bănică and Putinar. 1987, 2006.
- $n = 4$: Bănică and Putinar. M is a 4-dimensional complex manifold, $H^6(M)$ and $H^7(M)$ have no 2-torsion. 2006.
- $r = 2, n \leq 12$: Switzer, 1979.

$\mathbb{C}P^n$

- $\mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}P^n)$: Schwarzenberger condition.
- $r = 2, 3 \leq n \leq 6$: Atiyah and Rees (1976), Switzer(1979)

Main results

- M: an 8-dimensional closed oriented spin^c manifold.
- $\mathcal{C}_{14}\colon \mathrm{Vect}^4_{\mathbb{C}}(M) \to H^2(M) \times H^4(M) \times H^6(M) \times H^8(M)$: the map given by $\mathcal{C}_{14}(\eta) = (c_1(\eta), c_2(\eta), c_3(\eta), c_4(\eta))$,
- $\mathfrak{B}_M := \frac{\beta(H^5(M;\mathbb{Z}/2))}{\beta(\mathrm{Sq}^2(\rho_2(H^3(M))))}$,

where

- $\rho_2\colon H^i(M) \to H^i(M;\mathbb{Z}/2)$ is the mod 2 reduction.
- $\beta\colon H^i(M;\mathbb{Z}/2) \to H^{i+1}(M)$ is the Bockstein homomorphism.
- $\mathrm{Sq}^2\colon H^i(M;\mathbb{Z}/2) \to H^{i+2}(M;\mathbb{Z}/2)$ is the Steenrod square.

Main results

Theorem 1 (Y. 2022)

Let M be an 8-dimensional closed oriented spin^c manifold.
(A) For any even dimensional cohomology classes $u_i \in H^{2i}(M), 1 \leq i \leq 4$,

$$(u_1, u_2, u_3, u_4) \in \mathrm{Im}\ \mathcal{C}_{14}$$

if and only if they satisfy the following three conditions

(1) $\mathrm{Sq}^2(\rho_2(u_2)) = \rho_2(u_3 + u_1 u_2)$,

(2) $\langle u_4, [M] \rangle \equiv \langle p_1(M) u_2 - u_1^2 u_2 + u_1 u_3 - u_2^2, [M] \rangle \mod 3$,

(3) $\langle u_4, [M] \rangle \equiv$
$\langle -u_1^2 u_2 + u_1 u_3 + [2u_2^2 + p_1(M) u_2 - 3c^2 u_2]/4 + c(u_1 u_2 - u_3)/2, [M] \rangle$
mod 2,

where $p_1(M)$ is the first Pontrjagin class of M, $[M]$ is the fundamental class of M and $\langle \ \cdot \ , \ \cdot \ \rangle$ is the Kronecker product.

Main results

Theorem 1 (Y. 2022)

(B) For any $(u_1, u_2, u_3, u_4) \in \mathrm{Im}\, \mathcal{C}_{14}$, there is a bijection between

$$\mathcal{C}_{14}^{-1}(u_1, u_2, u_3, u_4) \quad \text{and} \quad \mathfrak{B}_M,$$

where $\mathcal{C}_{14}^{-1}(u_1, u_2, u_3, u_4)$ is the pre-image of (u_1, u_2, u_3, u_4) under the map $\mathcal{C}_{14}$. It follows that there is a one-to-one correspondence between

$$\mathrm{Vect}_{\mathbb{C}}^{4}(M) \quad \text{and} \quad \mathfrak{B}_M \times \mathrm{Im}\, \mathcal{C}_{14}.$$

Main results

Peterson (1959) tells us that if $H^6(M)$ has no 2-torsion, then the map $\mathcal{C}_{14}$ is injective. Obviously, as a corollary of Theorem 1, this statement can be generalized and strengthened as

Corollary

Let M be an 8-dimensional closed oriented spin^c manifold. Then the map $\mathcal{C}_{14}$ is injective if and only if $\mathfrak{B}_M = 0$.

Main results

- $I_* \colon \mathrm{Vect}^3_{\mathbb{C}}(M) \to \mathrm{Vect}^4_{\mathbb{C}}(M)$: the map given by $I_*(\alpha) = \alpha \oplus \epsilon$.
- $\mathcal{C}_{13} \colon \mathrm{Vect}^3_{\mathbb{C}}(M) \to H^2(M) \times H^4(M) \times H^6(M)$ the map given by $\mathcal{C}_{13}(\eta) = (c_1(\eta), c_2(\eta), c_3(\eta))$.

For a triple $u = (u_1, u_2, u_3) \in \mathrm{Im}\ \mathcal{C}_{13}$,

- $\mathfrak{T}_{M,u} := \frac{H^7(M)}{\{f^*(\gamma_7)+u_1 f^*(\gamma_5)+u_2 f^*(\gamma_3)+u_3 f^*(\gamma_1) \ \mid \ f \in [M,U]\}}$,

where

- $H^*(U) \cong \Lambda(\gamma_1, \gamma_3, \gamma_5, \gamma_7, \cdots)$.

Main results

Theorem 2 (Y. 2022)

Let M be an 8-dimensional closed oriented spin^c manifold.
(A) For any $\eta \in \mathrm{Vect}^4_{\mathbb{C}}(M)$, the necessary and sufficient condition for η to lie in the image of I_* is

$$c_4(\eta) = 0.$$

Therefore, for any cohomology classes $u_i \in H^{2i}(M; \mathbb{Z})$, $1 \leq i \leq 3$,

$$(u_1, u_2, u_3) \in \mathrm{Im}\ \mathcal{C}_{13} \quad \text{if and only if} \quad (u_1, u_2, u_3, 0) \in \mathrm{Im}\ \mathcal{C}_{14}.$$

(B) If $u = (u_1, u_2, u_3) \in \mathrm{Im}\ \mathcal{C}_{13}$, then $C_{13}^{-1}(u_1, u_2, u_3)$ is equivalent, as a set, to $\mathfrak{B}_M \times \mathfrak{T}_{M,u}$.

Main results

As applications, let us consider the classification of complex vector bundles over $\mathbb{C}P^4$.

- Set $t = -c_1(\gamma) \in H^2(\mathbb{C}P^4)$, where γ is the canonical line bundle,
- $H^*(\mathbb{C}P^4) = \mathbb{Z}[t]/\langle t^5 \rangle$,
- $c(\mathbb{C}P^4) = (1+t)^5$,
- $p_1(\mathbb{C}P^4) = 5t^2$, and we can take $c = c_1(\mathbb{C}P^4) = 5t$.
- $\mathfrak{B}_{\mathbb{C}P^4} = \mathfrak{T}_{\mathbb{C}P^4,u} = 0$ for any $u \in \mathrm{Im}\ \mathcal{C}_{13}$,

Main results

Corollary

The map

$$\mathcal{C}_{14}\colon \mathrm{Vect}^4_{\mathbb{C}}(\mathbb{C}P^4) \to H^2(\mathbb{C}P^4) \times H^4(\mathbb{C}P^4) \times H^6(\mathbb{C}P^4) \times H^8(\mathbb{C}P^4),$$

given by $\mathcal{C}_{14}(\eta) = (c_1(\eta), c_2(\eta), c_3(\eta), c_4(\eta))$*, is injective.*
Moreover, $(a_1 t, a_2 t^2, a_3 t^3, a_4 t^4) \in \mathrm{Im}\ \mathcal{C}_{14}$*, if and only if the integers* $a_i \in \mathbb{Z}$, $1 \le i \le 4$*, satisfy the following two conditions*

(1) $2a_4 \equiv a_2^2 + a_2 + a_1(a_1 a_2 - a_3) \mod 3,$

(2) $2a_4 \equiv a_2^2 + a_2 + a_1 a_2 - a_3 \mod 4.$ □

Main results

Corollary

The map

$$\mathcal{C}_{13}\colon \mathrm{Vect}^3_{\mathbb{C}}(\mathbb{C}P^4) \to H^2(\mathbb{C}P^4) \times H^4(\mathbb{C}P^4) \times H^6(\mathbb{C}P^4),$$

given by $\mathcal{C}_{13}(\eta) = (c_1(\eta), c_2(\eta), c_3(\eta))$*, is an injection.*
Furthermore, $(a_1 t, a_2 t^2, a_3 t^3) \in \mathrm{Im}\ \mathcal{C}_{13}$*, if and only if the integers* $a_i \in \mathbb{Z}$*,* $1 \le i \le 3$*, satisfy the following two conditions*

(1) $a_2^2 + a_2 + a_1(a_1 a_2 - a_3) \equiv 0 \mod 3,$

(2) $a_2^2 + a_2 + a_1 a_2 - a_3 \equiv 0 \mod 4.$ □

Main results

Remark

Schwarzenberger condition

Remark

Since the complex vector bundles with rank 2 over $\mathbb{C}P^4$ have been classified by Switzer, it follows that the classification of complex vector bundles over $\mathbb{C}P^4$ has been settled.

Proof of Theorem 1 (A)

- $\mathcal{C}_{24} = (c_2, c_3, c_4)\colon [M, BSU] \to H^4(M) \times H^6(M) \times H^8(M)$ be the map given by $\mathcal{C}_{24}(\eta) = (c_2(\eta), c_3(\eta), c_4(\eta))$ for any $\eta \in [M, BSU]$.

Theorem 1 (A')

Let M be an 8-dimensional closed oriented spin^c manifold. For any even dimensional cohomology classes $u_i \in H^{2i}(M), 2 \leq i \leq 4$,

$$(u_2, u_3, u_4) \in \mathrm{Im}\ \mathcal{C}_{24}$$

if and only if they satisfy the following three conditions

(1) $\mathrm{Sq}^2(\rho_2(u_2)) = \rho_2(u_3)$,

(2) $\langle u_4, [M]\rangle \equiv \langle p_1(M)u_2 - u_2^2, [M]\rangle \mod 3$,

(3) $\langle u_4, [M]\rangle \equiv \langle [2u_2^2 + p_1(M)u_2 - 3c^2u_2]/4 - cu_3/2, [M]\rangle \mod 2$.

Proof of Theorem 1 (A)

Lemma 1

For any $\eta \in [M, BSU]$, we must have

- $\langle c_4(\eta), [M] \rangle \equiv \langle p_1(M)c_2(\eta) - c_2^2(\eta), [M] \rangle \mod 3$,
- $\langle c_4(\eta), [M] \rangle \equiv \langle [2c_2^2(\eta) + p_1(M)c_2(\eta) - 3c^2c_2(\eta)]/4 - c\, c_3(\eta)/2, [M] \rangle$ mod 2.

Denote by $M^\circ := M - int(D^8)$ the space obtained from M by removing the interior of a small 8-disc in M. Let $p\colon M \to S^8$ be the map by collapsing M° to the basepoint, and $i\colon M^\circ \to M$ be the inclusion map.

Lemma 2

For any cohomology class $w \in H^8(M)$ with $\langle w, [M] \rangle \equiv 0 \mod 6$, there exists a stable complex vector bundle ξ' over S^8, such that $\xi = p^*(\xi')$ is a stable complex vector bundle over M satisfying $c_1(\xi) = c_2(\xi) = c_3(\xi) = 0$, and $c_4(\xi) = w$.

Proof of Theorem 1 (A)

- One direction is trivial.
- Another direction. Consider the map

$$\mathcal{C}_{23} = (c_2, c_3)\colon BSU \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$$

given by the universal Chern classes c_2 and c_3. It can be deduced easily from the homotopy sequence of this fiber space that

$$\pi_i(F) \cong \begin{cases} 0, & i \leq 7, i \neq 5; \\ \mathbb{Z}/2, & i = 5; \\ \pi_i(BSU), & i \geq 8. \end{cases}$$

Proof of Theorem 1 (A)

Therefore, the Postnikov resolution of the map $\mathcal{C}_{23}$ through dimension 9 is:

$$\begin{array}{ccccc} & & E & \longrightarrow & K(\mathbb{Z},9) \\ & \overset{h}{\nearrow} & \big\downarrow q & & \\ BSU & \xrightarrow[\mathcal{C}_{23}]{} & K & \xrightarrow{k} & K(\mathbb{Z}/2,6), \end{array}$$

where $K = K(\mathbb{Z},4) \times K(\mathbb{Z},6)$ and q is a principal fibration with fiber $K(\mathbb{Z}/2,5)$ and classifying map k. We will denote also by $k \in H^6(K;\mathbb{Z}/2)$ the class represented by the map k.

$$k = \mathrm{Sq}^2(\rho_2(l_4)) + \rho_2(l_6).$$

Proof of Theorem 1 (B)

We will divide the proof into the following three lemmas.
For any CW-complex X, denote by $V_0(X)$ the set of stable isomorphism classes of complex vector bundles over X with trivial Chern classes.
Obviously, $V_0(X)$ is a subgroup of $[X, BSU] \subset [X, BU]$.

Lemam 3

For any $(u_1, u_2, u_3, u_4) \in \mathrm{Im}\, \mathcal{C}_{14}$, there is a bijection between $\mathcal{C}_{14}^{-1}(u_1, u_2, u_3, u_4)$ and $V_0(M)$. Therefore, there is a one-to-one correspondence between $[M, BU]$ and $V_0(M) \times \mathrm{Im}\, \mathcal{C}_{14}$.

Lemma 4

The induced homomorphism $i^*\colon V_0(M) \to V_0(M^\circ)$ is bijective.

Proof of Theorem 1 (B)

Lemma 5

There is a one-to-one correspondence between $V_0(M^\circ)$ and

$$\mathfrak{B}_M = \frac{\beta(H^5(M;\mathbb{Z}/2))}{\beta(\mathrm{Sq}^2(\rho_2(H^3(M))))}.$$

Proof of Lemma 5

$$\begin{array}{ccccccc} & & & & [M^\circ, BSU] & & \\ & & & & \downarrow h_* & \searrow \mathcal{C}_{23*} & \\ [M^\circ, \Omega K] & \xrightarrow{(\Omega k)_*} & [M^\circ, K(\mathbb{Z}/2,5)] & \longrightarrow & [M^\circ, E] & \xrightarrow{q_*} & [M^\circ, K], \end{array}$$

where h_*, q_*, $\mathcal{C}_{23*}$ are the induced maps, $(\Omega k)_*$ is the induced homomorphism and where the bottom sequence is an exact sequence of sets.

Proof of Theorem 2

- $l_3 \colon BU(3) \to BU$,
- $l_{3*} \colon [M, BU(3)] \to [M, BU]$,
- $l_* \colon \mathrm{Vect}^3_{\mathbb{C}}(M) \to \mathrm{Vect}^4_{\mathbb{C}}(M)$.

The Postnikov resolution of the map l_3 through dimension 10 can be shown as

$$\begin{array}{ccccc} & & E & \longrightarrow & K(\mathbb{Z},10) \\ & \overset{h}{\nearrow} & \downarrow q & & \\ BU(3) & \xrightarrow[l_3]{} & BU & \xrightarrow{k} & K(\mathbb{Z},8). \end{array}$$

Here q is a principal fibration with fiber $K(\mathbb{Z},7)$ and k as the classifying map.