Smoothing homotopy theory

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Motivations

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Wish: Do homotopy theory in a convenient category with calculus tools.

Quillen's approach to homotopy theory – model categories

Def. A model structure on a bicomplete category $\mathcal{A}$ consists of three subcategories $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ together with the following axioms:

(1) Two out of three for $\mathcal{W}$
(2) Retraction for $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$
(3) Two liftings
(4) Two functorial factorizations

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Eg.
(1) Top with $\mathcal{W} =$ weak equivalences, $\mathcal{F} =$ Serre fibrations, $\mathcal{C}$ is defined by lifting.
(2) Top with $\mathcal{W} =$ homotopy equivalences, $\mathcal{C} =$ cofibrations, $\mathcal{F} =$ Hurewicz fibrations.
(3) sSets with $\mathcal{W} =$ weak equivalences, $\mathcal{F} =$ Kan fibrations, $\mathcal{C}$ is defined by lifting.

Model categories, II

Def. Given a model category $(\mathcal{A}; \mathcal{W}, \mathcal{C}, \mathcal{F})$, the localization category $\mathcal{W}^{-1}\mathcal{A} =: \mathit{Ho}(\mathcal{A})$ is called the homotopy category.

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Def. Given a model category $(\mathcal{A}; \mathcal{W}, \mathcal{C}, \mathcal{F})$, the localization category $\mathcal{W}^{-1}\mathcal{A} =: Ho(\mathcal{A})$ is called the homotopy category.

Def. An adjoint pair $F : \mathcal{A} \leftrightarrows \mathcal{B} : G$ between model categories is called a Quillen equivalence if
(1) $F(\mathcal{C}_{\mathcal{A}}) \subseteq \mathcal{C}_{\mathcal{B}}$ and $G(\mathcal{F}_{\mathcal{B}}) \subseteq \mathcal{F}_{\mathcal{A}}$.
(2) for any cofibrant C in $\mathcal{A}$ and any fibrant F in $\mathcal{B}$,
$C \to GF \in \mathcal{W}_{\mathcal{A}}$ iff the adjoint $FC \to B \in \mathcal{W}_{\mathcal{B}}$.

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(4) Typical example of Quillen equivalence:

$$|?| : sSets \leftrightarrows Top : S$$

Tools from differential topology

Smooth approximation theorem
Inverse function theorem (Implicit function theorem)
Sard's theorem
Whitney embedding and tubular neighborhood
Transversality and intersection theory
Morse theory and flows
Differential forms and de Rham theory
Riemannian metric and Hodge theory
Characteristic classes
(Co)bordism and surgery theory
Index theory
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All these are based on (smooth) manifolds (Hausdorff, second-countable, finite-dimensional).
But the category of such manifolds is usually too small to afford constructions in algebraic topology!

Tools from algebraic topology

CW approximation and cellular approximation
Bundle theory and classifying spaces
Spectra, infinite loop space, formal group laws and Brown representability
Hurewicz theorem
Postnikov and Whitehead towers
Obstruction theory
Freudenthal suspension theorem $\rightsquigarrow$ (un)stable homotopy theory
Steenrod operations
Various spectral sequences
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Q: Can we combine these tools in a suitable setting?

Generalizing smooth manifolds

Def (Souriau, 1980's). A diffeological space is a set X together with a specified family of maps $U \to X$ (called plots) for each open $U \subseteq \mathbb{R}^n$ and each n, such that for every open $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$:

(1) Every constant map $U \to X$ is a plot;

(2) If $U \to X$ is a plot and $V \to U$ is smooth, then $V \to U \to X$ is a plot;

(3) If $U = \bigcup_i U_i$ is an open cover and $U \to X$ is a map such that every restriction $U_i \to X$ is a plot, then $U \to X$ is a plot.

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Thm. There is an adjoint pair

$$\mathit{Diff} \leftrightarrows \mathit{Top}$$

Advantages of Diff

- Mfd is a full subcategory of Diff.
- Diff is bicomplete and cartesian closed.
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  $T_\theta := T^2/\mathbb{R}_\theta \cong \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$
  (1) T_θ has indiscrete D-topology
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  (3) The diffeomorphism group $\mathrm{Diff}(T_\theta) =$
  $$\begin{cases} (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ltimes T_\theta, & \text{if } \theta \text{ is quadratic irrational} \\ \mathbb{Z}/2\mathbb{Z} \ltimes T_\theta, & \text{otherwise.} \end{cases}$$

Advances in diffeology, I

- Differential forms and de Rham cohomology (Souriau, Iglesias-Zemmour)
- Various definitions of tangent spaces and tangent bundles (Hector, Iglesias-Zemmour, Christensen-W., Blohmann)
- Dimension (Iglesias-Zemmour)
- Diffeology and stratification (Iglesias-Zemmour, Gurer)
- Riemannian metric (Iglesias-Zemmour, Kuribayashi)
- Measure theory (Ahmadi-Magnot)
- Symplectic diffeology (Iglesias-Zemmour)
- Diffeology and orbifolds (Iglesias-Zemmour, Karshon, Zadka)
- Diffeology and NCG (Iglesias-Zemmour, Laffineur, Prato)
- ...

Advances in diffeology, II

- Homotopy theory (Iglesias-Zemmour, Christensen-W., Kihara, Haraguchi-Shimakawa)
- Rational homotopy (Kuribayashi)
- CW approximation (Iwase)
- Singular (co)homology (Kihara)
- Classifying spaces (Magnot-Watts, Christensen-W.)
- Various Čech cohomology (Iglesias-Zemmour, Krepski-Watts-Wolbert, Minichiello, Ahmadi)
  Čech-de Rham spectral sequence (Iglesias-Zemmour)
- . . .

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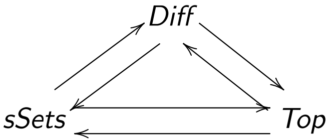
(3) As a conclusion, $\pi_1(T_\theta, x) = \mathbb{Z} \oplus \mathbb{Z}$. But $\pi_1^{Top}(T_\theta, x) = 0$. In other words, smooth approximation fails in general.

Smoothness vs continuity, II

Observations (Christensen-W.)



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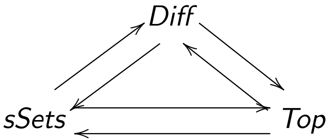

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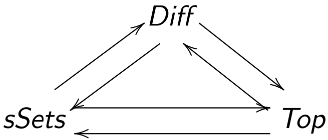

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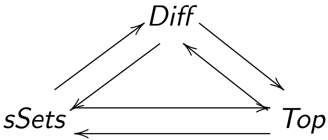

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Thm (Kihara, 2019). There is a model structure on Diff using bounded simplices, and there is a Quillen equivalence

$$\mathit{sSet} \leftrightarrows \mathit{Diff}$$

Smooth classifying spaces

Thm (Christensen-W.). Given any diffeological spaces X and F, any diffeological group G and any diffeological vector space V, there exists a diffeological space BG such that
(1) $[X, BG] \cong Prin_G^D(X)$ which is natural in X.
(2) $[X, BDiff(F)] \cong Bun_F^D(X)$ which is natural in X.
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Cor (Christensen-W.). BG is unique up to smooth homotopy equivalence.

Calculus tools (under construction)

Various tangent spaces and tangent bundles have been defined: Hector (with error), Iglesias-Zemmour (dual of cotangent), Christensen-W. (vector spaces and diffeological vector pseudo-bundles), Blohmann (diffeological spaces, elastic spaces, differential geometry)

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Diffeological vector spaces and homological algebra:
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Bott periodicity and characteristic classes?
completion and derivatives?

Thank you for your attention!