# The mapping class group of high dimensional manifolds

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# Differential manifolds

An $n$-dimensional *differential manifold* $M$ is a second countable, *locally euclidean*, Hausdorff space with *differentiable chart transformations*.

Example

1. spheres $S^n \subset \mathbb{R}^{n+1}$
2. closed orientable surfaces $S_g$ of genus $g$
3. non-singular real or complex varieties $X \subset \mathbb{P}^N$
4. homogeneous spaces $G/H$
5. products $M \times N$, connected sum $M \# N$

# Diffeomorphism group and mapping class group

The *diffeomorphism group*

$$\mathrm{Diff}(M) = \{\text{orientation preserving diffeomorphisms } f\colon M \to M\}$$

a topological group with the $C^\infty$ topology

The *mapping class group*

$$\mathrm{MCG}(M) = \mathrm{Diff}(M)/\text{isotopies} = \pi_0(\mathrm{Diff}(M))$$

a discrete group

Natural questions

1. determine the homotopy type of $\mathrm{Diff}(M)$
2. determine the discrete group $\mathrm{MCG}(M)$

# $\mathrm{Diff}(M)$ and $\mathrm{MCG}(M)$ in dimension $\leq 2$

1. $\mathrm{Diff}(S^1) \simeq SO(2)$ (exercise), $\mathrm{MCG}(S^1) = 0$
2. Smale (1958)

$$\mathrm{Diff}(S^2) \simeq SO(3), \ \mathrm{MCG}(S^2) = 0$$

3. $\mathrm{Diff}(T^2) \simeq T^2 \times \mathrm{SL}_2(\mathbb{Z})$, $\mathrm{MCG}(T^2) = \mathrm{SL}_2(\mathbb{Z})$
4. Earle-Eells (1969), Gramain (1973)

$$\mathrm{Diff}_{\mathrm{id}}(S_g) \simeq * \ \ (g \geq 2)$$

$\mathrm{MCG}(S_g) = \mathrm{Mod}_g$ the classical mapping class group

# $\mathrm{Diff}(M)$ and $\mathrm{MCG}(M)$ in dimension 3

1. Hatcher (1983)

$$\mathrm{Diff}(S^3) \simeq SO(4),\ \mathrm{MCG}(S^3) = 0$$

2. $\mathrm{Diff}(T^3) \simeq T^3 \times \mathrm{SL}_3(\mathbb{Z})$, $\mathrm{MCG}(T^3) = \mathrm{SL}_3(\mathbb{Z})$
3. Hatcher (1981)

$$\mathrm{Diff}(S^1 \times S^2) \simeq SO(2) \times SO(3) \times \Omega SO(3)$$

$$\mathrm{MCG}(S^1 \times S^2) = \pi_0 \Omega SO(3) = \pi_1 SO(3) = \mathbb{Z}/2$$

4. $M^3$ closed hyperbolic, $\pi_1(M) = \Gamma < \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$
   Gabai (2001)

$$\mathrm{Diff}(M) \simeq \mathrm{Isom}(M) = \mathrm{MCG}(M) = N(\Gamma)/\Gamma$$

# Examples of $\mathrm{MCG}(M)$ in dimension $\geq 5$

1. When $n \geq 5$, there is an isomorphism

$$\mathrm{MCG}(S^n) \xrightarrow{\cong} \Theta_{n+1}$$

$\Theta_{n+1} =$ the group of smooth structures on $S^{n+1}$.

Key ingredients:

- Surjectivity: Smale's $h$-cobordism theorem
- Injectivity: Cerf's pseudo-isotopy theorem

## Examples of $\mathrm{MCG}(M)$ in dimension $\geq 5$

2. Sato (1968): $2 \leq p < q$ , there is a short exact sequence

   $$1 \to FC_q^{p+1} \oplus \Theta_{p+q+1} \to \mathrm{MCG}(S^p \times S^q) \to \pi_p SO(q+1) \to 1$$

   $$FC_q^{p+1} = \{\text{framed knots } S^q \times D^{p+1} \hookrightarrow S^{p+q+1}\}/\text{isotopies}$$

3. Kreck (1978): $M = \#_g(S^n \times S^n)$ ($n \geq 3$), there are exact sequences

   $$1 \to \mathcal{T}(M) \to \mathrm{MCG}(M) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1$$

   $$0 \to \Theta_{2n+1} \to \mathcal{T}(M) \to \mathbb{Z}^{2g} \to 0$$

   Example

   $$1 \to \mathbb{Z}_{28} \to \mathrm{MCG}(S^3 \times S^3) \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \to 1$$

# finite/non-finite generation

**Theorem (Sullivan 1976)**

*If $\pi_1(M) = 0$ and $\dim M > 5$, then $\mathrm{MCG}(M)$ is commensurable to an arithmetic group. Thus $\mathrm{MCG}(M)$ is finitely presented.*

4. Hsiang-Sharpe (1976): when $n \geq 5$, there is a short exact sequence

$$1 \to \mathbb{Z}_2^\infty \oplus \binom{n}{2}\mathbb{Z}_2 \oplus \sum_{i=0}^{n} \binom{n}{i} \Theta_{i+1} \to \mathrm{MCG}(T^n) \to \mathrm{SL}_n(\mathbb{Z}) \to 1$$

In this talk we consider the mapping class group $\mathrm{MCG}(M)$

1. $M$ is a 3-dimensional complete intersection, e. g. the quintic Calabi-Yau;
2. $M$ is a compact (hyper)-Kähler manifold;
3. $M$ is a projective plane-like manifold.

## Complete intersections

A *complete intersection* with multidegree $\mathbf{d} = (d_1, \cdots, d_r)$

$$X(d_1, \cdots, d_r) = V(d_1) \cap \cdots \cap V(d_r)$$

is a transverse intersection of smooth hypersurfaces of degrees $d_1, \cdots, d_r$ in $\mathbb{CP}^{n+r}$. $X$ is an $n$-dim compact complex manifold.

**Example**

The hypersurface

$$X(n+2) = \{[z_0, \cdots, z_{n+1}] \mid z_0^{n+2} + \cdots + z_{n+1}^{n+2} = 0\} \subset \mathbb{CP}^{n+1}$$

is a Calabi-Yau $n$-fold.

1. $n = 1$, $X(3) = T^2$;
2. $n = 2$, $X(4)$ is a $K3$ surface;
3. $n = 3$, $X(5)$ is a quintic Calabi-Yau 3-fold.

# Mapping class group of 3d complete intersections

If $\dim_{\mathbb{C}} X = 3$, then by the Lefschetz Hyperplane theorem

$$\pi_1(X) = 0, \;\; H_2(X) = \mathbb{Z}, \;\; H_3(X) = \mathbb{Z}^{2g}.$$

**Theorem (Kreck-S. 2020)**

*There are short exact sequences*

$$1 \to \mathcal{T}(X) \to \mathrm{MCG}(X) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1$$

$$1 \to K(X) \to \mathcal{T}(X) \to (\mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g})/V_{\mathbf{d}} \to 1$$

*where* $V_{\mathbf{d}} \cong \mathbb{Z}^{2g}$, $K(X) = \mathrm{Center}(\mathrm{MCG}(X))$ *is a finite abelian group, all determined by the multidegree* $\mathbf{d}$*.*

**Theorem**

*The group $K(X)$ is determined by the multi-degree:*

$$K(M) \cong K(M)_{(2)} \times \mathbb{Z}/3^c \times \mathbb{Z}/7^e$$

*where*

$$c = \begin{cases} 1 & d \equiv 0 \pmod 3 \\ 0 & \text{otherwise} \end{cases} \qquad e = \begin{cases} 1 & d - l \equiv 0 \pmod 7 \\ 0 & \text{otherwise} \end{cases}$$

*The 2-primary part $K(M)_{(2)}$ is given in the following table*

| | $d$ odd | $d \equiv 2 \pmod 4$ | $d \equiv 4 \pmod 8$ | $d \equiv 0 \pmod 8$ |
|---|---|---|---|---|
| $l$ odd | $\mathbb{Z}_b$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $(\mathbb{Z}_2)^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_a \times \mathbb{Z}_b$ |
| $l$ even | $\mathbb{Z}_b$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $(\mathbb{Z}_2)^2 \times \mathbb{Z}_4$ | $\mathbb{Z}_2 \times \mathbb{Z}_a \times \mathbb{Z}_b$ |

$$a = \begin{cases} 1, & l \equiv \pm 1 \pmod 4 \\ 2, & l \equiv 2 \pmod 4 \\ 4, & l \equiv 0 \pmod 4 \end{cases} \qquad b = \begin{cases} 2, & l \equiv 1 \pmod 2 \\ 4, & l \equiv 0 \pmod 2 \end{cases}$$

## Example: the quintic Calabi-Yau 3-fold

**Corollary**

*For the quintic Calabi-Yau 3-fold*

$$X = \{[z_0, \cdots, z_5] \mid z_0^5 + \cdots + z_5^5 = 0\}$$

*we have*

$$d = 5, \quad g = 102, \quad l = -5,$$

*therefore* $\mathrm{Center}(\mathrm{MCG}(X)) = \mathbb{Z}_2$

$$\mathrm{MCG}(X)/\mathbb{Z}_2 = (\mathbb{Z}^{204} \oplus \mathbb{Z}_5^{204}) \rtimes \mathrm{Sp}(204, \mathbb{Z}).$$

**Theorem (Yu and Oguiso 2019)**

*The biholomorphic automorphism group*

$$\mathrm{Aut}(X) = \mathbb{Z}_5^4 \rtimes S_5.$$

# Group theoretical properties

For the mapping class group of surface $\mathrm{MCG}(F_g)$

1. $\mathrm{MCG}(F_g)$ is centerless
2. $\mathrm{MCG}(F_g)$ is perfect when $g \geq 3$, i. e. $H_1(\mathrm{MCG}(F_g)) = 0$
3. $\mathrm{MCG}(F_g)$ virtually torsion-free and residually finite.
4. The homomorphism $\mathrm{MCG}(F_g) \to \mathsf{Sp}(2g, \mathbb{Z})$ induces an isomorphism $H_2(\mathrm{MCG}(F_g)) \cong H_2(\mathsf{Sp}(2g, \mathbb{Z}))$ and the latter is isomorphic to $\mathbb{Z}$ when $g \geq 4$

**Theorem**

1. *The center of* $\mathrm{MCG}(X)$ *is* $K(X)$.
2. $\mathrm{MCG}(X)/K(X))$ *is virtually torsion-free and residually finite.*
3. $H_1(\mathrm{MCG}(X)) \cong H_1(\mathrm{Sp}(2g,\mathbb{Z}))$*, except for the case* $X$ *spin and* $d$ *even, where* $H_1(\mathrm{MCG}(X)) \cong \mathbb{Z}/2 \oplus H_1(\mathrm{Sp}(2g,\mathbb{Z}))$*. The group* $H_1(\mathrm{Sp}(2g,\mathbb{Z}))$ *is isomorphic to*

| $g = 1$ | $g = 2$ | $g \geq 3$ |
|---|---|---|
| $\mathbb{Z}/12$ | $\mathbb{Z}/2$ | 0 |

4. *There is a surjective homomorphism* $H_2(\mathrm{MCG}(X)) \to H_2(\mathrm{Sp}(2g,\mathbb{Z}))$*. When* $g \geq 3$ *we have* $H_2(\mathrm{MCG}(M = X);\mathbb{Q}) \cong \mathbb{Q}^2$*.*

# Monodromy representation of the family of hypersurfaces

- $\mathcal{U}_d \subset \mathbb{P}(H^0(\mathbb{C}\mathrm{P}^4; \mathcal{O}(d)))$ the space of smooth hypersurfaces of degree $d$ in $\mathbb{C}\mathrm{P}^4$
- $X(d)$ the Fermat hypersurface, a base point in $\mathcal{U}_d$
- $\mathcal{X}_d \to \mathcal{U}_d$ the universal family of smooth hypersurfaces of degree $d$ in $\mathbb{C}\mathrm{P}^4$
- monodromy representation $\alpha\colon \pi_1(\mathcal{U}_d, X(d)) \to \mathrm{MCG}(X(d))$

**Theorem (Randal-Williams 2024)**

*Assume $d \geq 3$, there is a short exact sequence*

$$0 \to K \to \mathrm{Im}(\alpha) \to \mathrm{Aut}(\pi_3(X(d)), \lambda, \mu) \to 1.$$

# Torelli group of (hyper)-Kähler manifolds

There is a short exact sequence

$$1 \to \mathcal{T}(M) \to \mathrm{MCG}(M) \to \mathrm{Aut}(H^*(M))$$

where $\mathcal{T}(M)$ is called the *Torelli group* of $M$.

**Theorem (Kreck-S.)**

*For any $n \geq 3$ there are compact Kähler manifolds of complex dimension $n$ with infinite Torelli group.*

Two families of hyperkähler manifolds: Let $S$ be a compact complex algebraic surface, $S^{(m)} = S \times \cdots \times S/\mathfrak{S}_m$ be its $m$-fold symmetric product. There exists a canonical resolution $\varepsilon \colon S^{[m]} \to S^{(m)}$ which is a smooth complex $2m$-dimensional manifold.

- $S = K$ a $K3$-surface, then $K^{[m]}$ is a hyperkähler manifold;
- $S = T$ a torus, then the non-trivial factor in the de Rham decomposition of the universal cover of $T^{[m]}$ is a hyperkähler manifold of complex dimension $2m - 2$, denoted by $K_{m-1}$.

**Theorem (Kreck-S.)**

1. *The Torelli group of $K_2$ is infinite.*
2. *The Torelli group of $K^{[2]}$ is finite.*

# Projective plane-like manifolds

$M$ closed smooth, *looks like a projective plane* if

$$\pi_1(M) = 0, \quad H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Example $\quad M = \mathbb{C}\mathrm{P}^2, \mathbb{H}\mathrm{P}^2, \mathbb{O}\mathrm{P}^2$

Kramer-Stolz (2007): Diffeomorphism classification of $\mathbb{H}\mathrm{P}^2$- and $\mathbb{O}\mathrm{P}^2$-like manifolds

- $\dim M = 8$

$$\hat{A}(M_t) = -\frac{t(t+1)}{7 \cdot 8}, \quad t \equiv 0, 7, 48, 55 \pmod{56}$$

- $\dim M = 16$

$$\hat{A}(M_t) = -\frac{t(t+1)}{127 \cdot 128}, \quad t \equiv 0, 127, 16128, 16255 \pmod{16256}$$

# $\hat{A}$-genus and $\alpha$-invariant

The $\hat{A}$-genus is a ring homomorphism

$$\hat{A}\colon \Omega^{\mathrm{spin}}_{4*} \to \mathbb{Z}$$

The $\alpha$-invariant is a generalization of $\hat{A}$

$$\hat{\mathscr{A}}\colon \Omega^{\mathrm{spin}}_{*} \to KO^{-*}(\mathrm{pt})$$

| $*$ (mod 8) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $KO^{-*}(\mathrm{pt})$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

**Theorem**

*A simply-connected closed spin manifold $M$ of dimension $\geq 5$ admits a positive scalar curvature metric if and only if $\hat{\mathscr{A}}(M) = 0$.*

# Mapping class group of projective plane-like manifolds

**Theorem (S.-Wang Wei 2023)**

1. *Let $M$ be an $\mathbb{HP}^2$-like manifold, then $\mathrm{MCG}(M) \cong \mathbb{Z}_2$. If $\hat{A}(M)$ is even, there is an isomorphism*

$$\mathrm{MCG}(M) \to \mathbb{Z}/2, \quad f \mapsto \hat{\mathscr{A}}(M_f).$$

2. *Let $M$ be an $\mathbb{OP}^2$-like manifold, then $\mathrm{MCG}(M) \cong \mathbb{Z}_2^3$*

**Corollary**

*For a positive scalar curvature metric on $\mathbb{HP}^2$, any isometry is isotopic to the identity.*

*Thank You*