An overview of unstable homotopy decompositions

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GBU Conference on Algebraic Topology

Jun. 20-21, 2025

A Basic Question

- X = polyhedra, CW-complex, manifold, variety, etc
- Homotopy group $\pi_n(X) = [S^n, X]$

Question:

How to compute $\pi_*(X)$?

- It is **impossible** to get a complete answer!
- e.g. $\pi_n(S^m)$?

Homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

from wiki

Global characterizations of homotopy groups

- homotopy exponent problem
- exponential growth in homotopy groups
- a periodic problem of homotopy groups

Localization

X = 1-connected CW-complex.

- Algebraic localization:

$$\begin{array}{ccccc} & & \mathbb{Z}_{(p)} & & \\ & \nearrow & & \searrow & \\ \mathbb{Z} & & & & \mathbb{Q} \\ & \searrow & & \nearrow & \\ & & \mathbb{Z}_{(\frac{1}{p})} & & \end{array}$$

- Homotopy realization:

$$\begin{array}{ccccc} & & X_{(p)} & & \\ & \nearrow & & \searrow & \\ X & & & & X_{\mathbb{Q}} \\ & \searrow & & \nearrow & \\ & & X_{(\frac{1}{p})} & & \end{array}$$

Homotopy exponents

Fred Cohen
(1945-2022)

- (Cohen-Moore-Neisendorfer, '79)
  $p^n \cdot \pi_*(S^{2n+1})_{(p)} = 0$ if $p \geq 5$
- exponent $\exp_p(S^{2n+1}) = p^n$

- (50s-00s) James, Barratt, Moore, Gray, Selick, Cohen, Neisendorfer, Wu, Theriault, etc
- method: loop space decomposition

Loop space decomposition

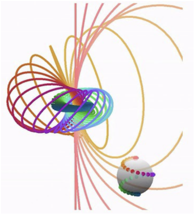
- $X = (X, *) =$ a based CW-complex
- Loop space $\Omega X := \mathrm{Map}_*(S^1, X)$
- loop space $\simeq$ top group

- Loop decomposition: $\Omega X \simeq Y \times Z$
- $\pi_{*+1}(X) \cong \pi_*(\Omega X) \cong \pi_*(Y) \oplus \pi_*(Z)$

Hopf fibration ('31, Hopf)

- $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$
- $S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$
- $S^7 \longrightarrow S^{15} \longrightarrow S^8$

- $\Omega\mathbb{C}P^n \simeq S^1 \times \Omega S^{2n+1}$
- $\Omega\mathbb{H}P^n \simeq S^3 \times \Omega S^{4n+3}$
- $\Omega S^8 \simeq S^7 \times \Omega S^{15}$



from youtube

Lie groups ('53, Serre)

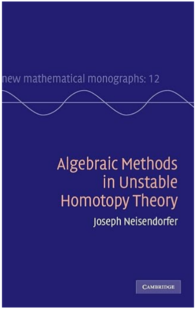
- $p > 2n-1$, $SO(2n+1) \simeq_p S^3 \times \cdots \times S^{4n-1}$,
- $p > 2n-3$, $SO(2n) \simeq_p S^3 \times \cdots \times S^{4n-5} \times S^{2n-1}$
- $p > n-1$, $SU(n) \simeq_p S^3 \times \cdots \times S^{2n-1}$
- $p > 2n-1$, $Sp(n) \simeq_p S^3 \times \cdots \times S^{4n-1}$
- $p > 5$, $G_2 \simeq_p S^3 \times S^{11}$
- $p > 11$, $F_4 \simeq_p S^3 \times S^{11} \times S^{15} \times S^{23}$
- $p > 11$, $E_6 \simeq_p S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}$
- $p > 17$, $E_7 \simeq_p S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}$
- $p > 29$, $E_8 \simeq_p S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}$

Fact: $\Omega BG \simeq G$.

Three methods in historical order

- decompositions via homological information;
- decompositions via idempotents;
- decompositions via Cube techniques.

Decompositions via homological information



574 pages on CMN

- '55, James suspension splitting, $\Sigma\Omega\Sigma X$
- '72, Hilton-Milnor theorem, $\Omega\Sigma(X \vee Y)$
- '79, CMN decomposition, Moore spaces

- exponent problem
- Moore Conjecture
- Barratt-Cohen Conjecture
- exponential growth in torsion homotopy groups

Moore Conjecture

Moore conjecture; '70s

For a simply connected finite complex Z, the following are equivalent

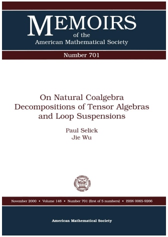
- at any prime p

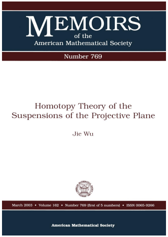
$$p^N \cdot (p \text{ torsions of } \pi_*(Z)) = 0$$

  for sufficiently large N;
- $\pi_*(Z) \otimes \mathbb{Q}$ is finite dimensional.

The conjecture is OPNE...

Decompositions via idempotents





- '77- , refined decomposition of Lie groups, Mimura, Nishida, Toda, Theriault
- '00- , functorial decomposition, Selick, Wu, Theriault

- exponent problem for Lie groups
- computation of homotopy groups $\Sigma^{?}\mathbb{R}P^2$, Moore spaces, etc.
- a periodic property of homotopy groups
- the interaction with modular representation theory

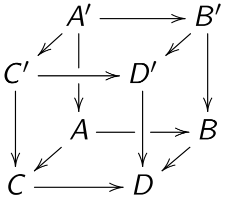
Decompositions via Cube techniques

From 21', Huang and Theriault use Cube techniques to decompose Ω(geometric objects)

- highly connected Poincaré duality complexes
- manifolds stabilized by projective spaces
- surgery
- blow up
- open book
- polyhedral products

- homotopy groups and their exponential growth
- evidences for Moore conjecture
- Gromov-Vigué-Poirrier Conjecture
- new conjecture and open problems

A Cube tool



Mather's Cube Lemma; '76

Suppose in the above homotopy commutative diagram

- the vertical faces are homotopy pullbacks, and
- the bottom face is a homotopy pushout.

Then the top face is a homotopy pushout.

Example

- $M=$ $(n-1)$-connected $2n$-manifold, $n \geq 2$
- $H^n(M) \cong \mathbb{Z}^{\oplus k}$, $k \geq 2$

Theorem (Beben-Theriault; '14)

When $n \neq 2, 4, 8$,

$$\Omega M \simeq \Omega(S^n \times S^n) \times \Omega\big(J_n \vee (J_n \wedge \Omega(S^n \times S^n))\big)$$

When $n = 2$,

$$\Omega M \simeq S^1 \times \Omega(S^2 \times S^3) \times \Omega\big(J_2 \vee (J_2 \wedge \Omega(S^2 \times S^3))\big)$$

where $J_n = \bigvee_{k-2} S^n$, $J_2 = \bigvee_{k-2} (S^2 \vee S^3)$.

Concrete studies

Studies on concrete cases of k-connected m-manifolds:

- $k = n - 1$, $m = 2n$: Beben-Theriault ('14);
- $k = n - 1$, $m = 2n + 1$: Beben-Wu ('15), Huang-Theriault ('21);
- $k = n - 2$, $m = 2n$ $(n > 3)$: Chenery ('22);
- $k = 1$, $m = 5$: Beben-Theriault ('18), Theriault ('20);
- $k = 1$, $m = 6$: Huang ('21).

The loop decompositions of these concrete manifolds support Moore conjecture.

Example ('22; Duan-H-Theriault)

$$\Omega(\mathbb{C}P^{2n}\#\mathbb{C}P^{2n}) \simeq S^1 \times S^1 \times \Omega S^3 \times \Omega S^{4n-1},$$
$$\Omega(\mathbb{C}P^{2n}\#\mathbb{H}P^{n}) \simeq S^1 \times S^3 \times \Omega S^5 \times \Omega S^{4n-1},$$
$$\Omega(\mathbb{C}P^{8}\#\mathbb{O}P^{2}) \simeq S^1 \times S^7 \times \Omega S^9 \times \Omega S^{15},$$
$$\Omega(\mathbb{C}P^{2n+1}\#\mathbb{C}P^{2n+1}) \simeq_{\{\frac{1}{2}\}} S^1 \times S^1 \times \Omega S^3 \times \Omega S^{4n+1},$$
$$\Omega(\mathbb{H}P^{n}\#\mathbb{H}P^{n}) \simeq_{\{\frac{1}{2},\frac{1}{3}\}} S^3 \times S^3 \times \Omega S^7 \times \Omega S^{4n-1},$$
$$\Omega(\mathbb{H}P^{4}\#\mathbb{O}P^{2}) \simeq_{\{\frac{1}{2},\frac{1}{3}\}} S^3 \times S^7 \times \Omega S^{11} \times \Omega S^{15}$$

Free loop space

- Let X be a topological space.
- Free loop space: $\mathcal{L}X = \mathrm{Map}(S^1, X)$
  (cf. Based loop space: $\Omega X = \mathrm{Map}_*(S^1, X)$)

- Free loop space matters in geometry, topology and mathematical physics, etc
- Chaos-Sullivan '99: string topology

- $M =$ Riemannian manifold
- (Gromov '78; Ballmann-Ziller '82) Betti numbers of $\mathcal{L}M$ gives a lower bound of the number of closed geodesics on M.

Gromov-Vigué-Poirrier conjecture

Classical rational dichotomy; Félix-Halperin-Thomas-Lemaire, '89

Any simply-connected finite CW-complex X is either:

- *rationally elliptic, that is,* $\pi_*(X) \otimes \mathbb{Q}$ *is finite dimensional, or else*
- *rationally hyperbolic, that is,* $\pi_*(X) \otimes \mathbb{Q}$ *grows exponentially.*

Gromov-Vigué-Poirrier conjecture; '78, '84

If X is hyperbolic, then $\dim H_\ell(\mathcal{L}X; \mathbb{Q})$ grows exponentially.

- (Vigué-Poirrier, '84), (Lambrechts, '01),
- (Félix-Halperin-Thomas, '13, '17), (H-Theriault, '21).

Moore Conjecture

Moore conjecture, elliptic version

The following are equivalent for a 1-con finite CW-complex X:

(a) X is rationally elliptic;

(b) $\exp_p(X) < \infty$ for some prime p;

(c) $\exp_p(X) < \infty$ for all primes p.

Moore conjecture, hyperbolic version

Then the following are equivalent for a 1-con finite CW-complex X:

(a) X is rationally hyperbolic;

(b) $\exp_p(X) = \infty$ for some prime p;

(c) $\exp_p(X) = \infty$ for all primes p.

H-Theriault Conjecture

Local hyperbolic

A *CW*-complex X is called *mod-p^r hyperbolic* if the number of $\mathbb{Z}/p^r\mathbb{Z}$-summands in $\pi_*(X)$ has exponential growth.

H-Theriault Conjecture, '22

If a 1-con finite *CW*-complex X is rationally hyperbolic then it is mod-p^r hyperbolic for all primes p and positive integers r.

A book by H-Theirault, coming soon...

- Book title: Unstable Homotopy Decompositions
  in Series in Algebraic and Differential Geometry, Series Editor Phillip A Griffiths, World Scientific Publishing
- https://doi.org/10.1142/14126
- ISBN: 978-981-98-0613-3 (hardcover)

- 358 pages + xvi
- the three methods
- various applications
- suspension splittings
- 40 open problems

Homotopy effect of surgery

(H-Theriault, '22): two types of surgery

Problems:

- Study the homotopy effect of other types of surgeries
- Which types of surgery will preserve rational ellipticity or hyperbolicity? Which will change them?
- Which types of surgery will preserve or change mod-p^r hyperbolicity, or homotopy exponents?

The problems are designed for attacking Moore conjecture

Barratt Conjecture

Barratt Conjecture

Let X be a co-H-space and suppose that the degree p^r map on X is null homotopic. Then $p^{r+1} \cdot \pi_*(\Sigma^2 X) = 0$.

The Barratt-Cohen Conjecture

Let $f : \Sigma^2 X \to Z$ be a map such that $p^r[f] = 0$ in the group $[\Sigma^2 X, Z]$. Then

$$\Omega^2 f : \Omega^2 \Sigma^2 X \to \Omega^2 Z$$

has order at most p^{r+1} in the group $[\Omega^2 \Sigma^2 X, \Omega^2 Z]$.

Thanks very much