Spectral Gap of Dirac Operator on Spin Manifold with applications to nonlinear problems

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Contents

From Theoretical Physics

- The energy-momentum relation of a free relativistic particle:

$$E^2 = c^2|p|^2 + m^2c^4.$$

- The usual identification

$$p \leftrightarrow -i\hbar\nabla.$$

- Goal: Find a self-adjoint operator D_c satisfying

$$(D_c)^2 = -c^2\hbar^2\Delta + m^2c^4.$$

From Theoretical Physics

- **Dirac's solution:**

$$D_c = -ic\hbar\alpha \cdot \nabla + mc^2\beta,$$

where $\alpha \cdot \nabla = \sum_{k=1}^{3} \alpha_k \partial_k$, and $\partial_k = \frac{\partial}{\partial x_k}$, α_1, α_2, α_3 and β are 4×4 Pauli-Dirac matrices

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From Theoretical Physics

- **Free Dirac equation:**

$$i\hbar\frac{\partial}{\partial t}\Psi(t,x) = D_c\Psi(t,x).$$

Question. What does that mean?

- $\Psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ is the wave function of the Dirac particle.
- Dirac particles: spin $1/2$, massive fermions (leptons, quarks).
- Anti-particle, spin up or down.
- Probability

$$\mathbf{P} = \int_V P(x,y,z)dxdydz = \int_V |\Psi(t,x)|^2 dx.$$

From Physics to Mathematics

Question. How to generalize the Dirac operator to $\mathbb{R}^{n+1}$?

Idea: Since $D_c = -ic\hbar\alpha \cdot \nabla + mc^2\beta$, we only need to generalize the Dirac matrices $(\{\alpha_k\}_{k=1}^3, \beta)$.

Definition (Dirac Matrices)

For $(n+1)$ dimensional space, $(\{\alpha_k\}_{k=1}^n, \beta))$ is an $(n+1)$-tuple of Dirac matrices if

- β, α_k are symmetric $N \times N$ matrices.
- $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$, $\alpha_i\beta + \beta\alpha_i = 0$, $\beta^2 = 1$, for $i, j = 1, ..., n$.

The smallest possible dimension N of the spinor space to admit Dirac matrices is $2^{[\frac{n+1}{2}]}$.

Reference: B. Thaller, The Dirac Equation, *Theoretical and Mathematical Physics*, Springer Berlin, 1992.

From Physics to Mathematics

Proposition (Existence and Structure of Dirac Matrices)

There is an $(n+1)$-tuple of Dirac matrices in $M_N(\mathbb{C})$ when $N = 2^{[\frac{n+1}{2}]}$. Moreover, we have $(\{\alpha_k\}_{k=1}^n, \beta)$ has the form

$$\alpha_k = \begin{pmatrix} 0 & a_k \\ a_k^* & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{pmatrix},$$

where the a_k are $\frac{N}{2} \times \frac{N}{2}$ matrices (which are Hermitian if n is odd).

- **Examples of Low Dimension.**

n=1 $N = 2$, $\alpha_1 = \sigma_1$, $\beta = \sigma_3$.

n=2 $N = 2$, $\alpha_1 = \sigma_1$, $\alpha_2 = \sigma_2$, $\beta = \sigma_3$.

n=3 $N = 4$, $\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, where $1 \le j \le 3$.

From Physics to Mathematics

- **Examples of High Dimension.**
Bosonic String Theory: $n = 25$. Superstring Theory: $n = 9$.
M-Theory: $n = 10$.
- **Observation.** For $n = 3$, we also use the gamma matrices:

$$\gamma^0 = \beta,\ \gamma^0\gamma^j = \alpha_j,\ 1 \leq j \leq 3.$$

The Clifford relation: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}I$. This is related to Clifford algebra.
- **Clifford Algebra** Let $e_1, \ldots, e_n$ be an orthonormal basis of $(\mathbb{R}^n, g)$. Then the (finite dimensional!) associative algebra

$$Cl\left(\mathbb{R}^n\right) := \bigotimes \mathbb{R}^n / \left\{e_i \cdot e_j + e_j \cdot e_i = 0, e_i^2 = -1\right\}$$

is called the Clifford algebra of $\mathbb{R}^n$. $Cl^{\mathbb{C}}\left(\mathbb{R}^n\right)$ denotes its complexification.

A brief review of Clifford Algebras

Definition (Clifford Algebra)

$V = \mathbb{K}^n$, g a nondegenerate bilinear form on V. The Clifford algebra is defined by

$$Cl(V,g) := T(V)/I(V,g),$$

where $T(V)$ is the tensor algebra of V, $I(V,g)$ is the ideal generated by all elements of the form $x \otimes x + g(x,x)1$, for $x \in V$.

Remark (1) $Cl(V,g)$ is generated by the relation

$$x \cdot y + y \cdot x = -2g(x,y)1, \quad x, y \in V.$$

(2) $\{e_{i_1} \cdot \ldots \cdot e_{i_k} : 1 \le i_1 < \ldots < i_k \le n, 0 \le k \le n\}$ is a basis of $Cl(V,g)$. Thus, $\dim Cl(V,g) = 2^n$.

From Physics to Mathematics

Theorem (Representation of Clifford Algebra)

There exists a unique representation of smallest dimension of the algebra $Cl^{\mathbb{C}}(\mathbb{R}^n)$ on a complex vector space Δ_n :

$$Cl^{\mathbb{C}}(\mathbb{R}^n) \longrightarrow \mathrm{End}(\Delta_n), \quad \dim \Delta_n = 2^{[n/2]}.$$

Δ_n : space of (Dirac) spinors.

- **Example.** The representation of $Cl_2^{\mathbb{C}} := Cl^{\mathbb{C}}(\mathbb{R}^2)$ is given by

$$Cl_2^{\mathbb{C}} \to M_2(\mathbb{C})$$

$$1 \to E, e_1 \to g_1, e_2 \to g_2, e_1 \cdot e_2 \to -iT.$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

From Mathematics

- The $\mathrm{Spin}(n)$ group is a two-fold covering of $\mathrm{SO}(n)$ and can be realized in $Cl\,(\mathbb{R}^n)$,

$$\mathrm{Spin}(n) = \{x_1 \cdot \ldots \cdot x_{2l}, x_i \in \mathbb{R}^n \text{ and } |x_i| = 1\}.$$

- Every vector $x \in \mathbb{R}^n$ acts on Δ_n by an endomorphism:

$$\mathbb{R}^n \times \Delta_n \ni (x, \psi) \longmapsto x \cdot \psi \in \Delta_n \quad \text{Clifford multiplication}$$
$$\mu : \mathbb{R}^n \otimes \Delta_n \longrightarrow \Delta_n.$$

- The $\mathrm{Spin}(n)$-representation $\mathbb{R}^n \otimes \Delta_n$ splits into

$$\mathbb{R}^n \otimes \Delta_n = \Delta_n \oplus \ker(\mu)$$

From Mathematics

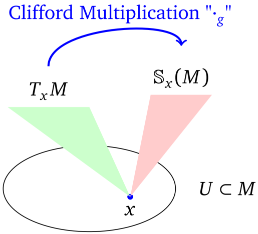
- **Idea:** Attach a copy of Δ_n to every point x of a Riemannian manifold (M, g) :

Tangent bundle:

$$T(M) = \bigcup_{x \in M} T_x M$$

Spinor bundle:

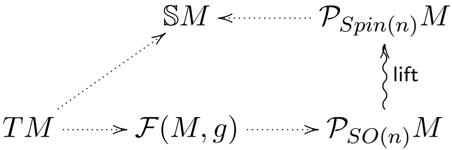
$$\mathbb{S}(M) = \bigcup_{x \in M} \Delta_n(x)$$



- **Idea:** Denote by $\mathcal{F}(M, g)$ the oriented frame bundle. M admits a spin structure iff its $\mathrm{SO}(n)$-principal bundle $\mathcal{P}_{SO(n)}M$ admits a reduction $\mathcal{P}_{Spin(n)}M \to \mathcal{P}_{SO(n)}M$ to the group $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

From Mathematics

- **Idea:**



- **Spinor bundle** $\mathbb{S}M = \mathcal{P}_{Spin(n)}M \times_\mu \Delta_n$.
- **Section** A section $\psi \in \Gamma(\mathbb{S}M)$ is locally given by

$$\psi|_U = [\tilde{s}, \sigma],$$

where $\tilde{s} \in \Gamma(\mathcal{P}_{Spin(n)}M)$, $U \subset M$, $\sigma : U \to \Delta_n$.

From Mathematics

- The first Stiefel-Whitney class $w_1(M) \in H^1(M, \mathbb{Z}_2)$ vanishes if and only if M is orientable.
- The second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes if and only if M admits a spin structure.
- There is a non-canonical bijection

$$\{[\text{spin structures}]\ \} \leftrightarrow \text{Hom}(\pi_1(M), \mathbb{Z}_2) \leftrightarrow H^1(M, \mathbb{Z}_2).$$

- $\mathbb{H}$, $S^n (n \geq 2)$ are spin manifolds with a unique spin structure. S^1 admits 2 different spin structures. T^n admits 2^n different spin structures.

From Mathematics

- The Clifford multiplication on $\mathbb{S}M$ is the fiberwise action given by

$$\mu : TM \otimes \mathbb{S}M \to \mathbb{S}M$$
$$X \otimes \psi \to X \cdot \psi,$$

span

where $X = [\tilde{s}, \alpha]$, $X \cdot \psi := [\tilde{s}, \alpha \cdot \sigma]$, $\alpha \cdot \sigma$ is the Clifford multiplication on Δ_n.

$$TM \cong \mathcal{P}_{Spin(n)}M \times_{Ad} \mathbb{R}^n.$$

From Mathematics

The lift from a section $s \in \Gamma_U(\mathcal{P}_{SO(n)}M)$ to $\tilde{s} \in \Gamma_U(\mathcal{P}_{Spin(n)}M)$

$$\begin{array}{ccc} & & \mathcal{P}_{Spin(n)}M \\ & \overset{\tilde{s}}{\nearrow} & \downarrow \eta \\ U \subset M & \xrightarrow{\ s\ } & \mathcal{P}_{SO(n)}M \end{array}$$

induces a connection 1-form on $\mathcal{P}_{Spin(n)}M$

$$\begin{array}{ccccc} & & T\mathcal{P}_{Spin(n)}M & \xrightarrow{\ \tilde{\omega}\ } & \mathfrak{spin}_n \\ & \overset{\tilde{s}_*}{\nearrow} & \downarrow \eta & & \downarrow Ad_* \\ TU \subset TM & \xrightarrow{\ s_*\ } & T\mathcal{P}_{SO(n)}M & \xrightarrow{\ \omega\ } & \mathfrak{so}_n \end{array}$$

From Mathematics

- **Spinorial Covariant derivative** Take an orthonormal basis $\sigma_1, ..., \sigma_N$ of Δ_n to get a local section $\{\psi_\alpha\}_{1\leq\alpha\leq N}$ by

$$\psi_\alpha := [\tilde{s}, \sigma_\alpha] \in \Gamma_U(\mathbb{S}M).$$

Then the spinorial covariant derivative is given locally by

$$\nabla\psi_\alpha = \frac{1}{4}\sum_{i,j=1}^{n} g(\nabla e_i, e_j)e_i \cdot e_j \cdot \psi_\alpha.$$

From Mathematics

- **Dirac operator** The Dirac operator is the composition of the covariant derivative acting on sections of $\mathbb{S}M$ with the Clifford multiplication

$$\mathcal{D} := \mu \circ \nabla.$$

Locally, we have

$$\begin{aligned} \mathcal{D}: \quad \Gamma(\mathbb{S}M) &\xrightarrow{\nabla} \Gamma(T^*M \otimes \mathbb{S}M) \xrightarrow{\mu} \Gamma(\mathbb{S}M) \\ \psi \quad &\longrightarrow \sum_{i=1}^{n} e_i^* \otimes \nabla_{e_i}\psi \longrightarrow \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi \end{aligned}$$

span

- Dirac operator is a first order differential operator which is elliptic and formally self-adjoint.

From Mathematics

- **Example 1.** Let $M = \mathbb{R}^n$, $\mathbb{S}M = \mathbb{R}^n \times \mathbb{C}^N$, then every spinor field $\psi \in \Gamma(\mathbb{S}M)$ is in fact a map $\psi : \mathbb{R}^n \to \mathbb{C}^N$, and the Dirac operator is given by

$$\mathcal{D} = \sum_{i=1}^{n} e_i \cdot \partial_i = \sum_{i=1}^{n} \mu(e_i)\partial_i,$$

where $\mu(e_i) \in M_N(\mathbb{C})$ satisfies $\mu(e_i)\mu(e_j) + \mu(e_j)\mu(e_i) = 2\delta_{ij} I_N$.
(This is in fact the Dirac matrices)

- **Example 2.** Let $M = \mathbb{R}^2$, (e_1, e_2) be the orthonormal basis of $\mathbb{R}^2$. The complex volume element $\omega_{\mathbb{C}} := i^{[\frac{n+1}{2}]} e_1 \cdot \ldots \cdot e_n = i e_1 \cdot e_2$. Then $\Delta_2 = \Delta_2^+ \oplus \Delta_2^- \cong \mathrm{span}_{\mathbb{C}}\{e_1, e_2\}$, where

$$\Delta_2^{\pm} = \frac{1}{2}(1 \pm \omega_{\mathbb{C}}) \cdot \Delta_2 \cong \mathrm{span}_{\mathbb{C}}\{1 \pm e_2\}.$$

From Mathematics

Then each spinor field $\psi \in \Gamma(\mathbb{S}M)$ is given by two complex functions $f, g : \mathbb{R}^2 \to \mathbb{C}$, such that

$$\psi = f(1 + e_2) + g(1 - e_2).$$

The Dirac operator becomes

$$\begin{aligned} \mathcal{D}\psi &= (e_1 \cdot \partial_1 + e_2 \cdot \partial_2)\,(f(1 + e_2) + g(1 - e_2)) \\ &= (1 + e_2)\,(i\partial_1 + \partial_2)\,g + (1 - e_2)\,(i\partial_1 - \partial_2)\,f \qquad \text{span} \\ &= 2i\partial_z g(1 + e_2) + 2i\partial_{\bar{z}} f(1 - e_2). \end{aligned}$$

That is

$$\mathcal{D} = 2i \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}$$

in the basis $\{1 + e_2, 1 - e_2\}$ of Δ_2. Hence the Dirac operator can be considered as a generalization of the Cauchy-Riemann operator.

Dirac operator and Laplace operator

- Dirac operator enjoys analogous properties to the Laplace-Beltrami operator:
  - conformally covariant
  - self-adjoint
  - discrete eigenvalues of finite multiplicity
- Difference:
  - Dirac operator is a first order differential operator
  - Dirac operator acts on spinors (which are complex vectors)
  - the spectrum of Dirac operator accumulates both $+\infty$ and $-\infty$

Supplement to the Similarity

Case I. Consider $Au = W(x)|u|^{p-2}u$, $W \geq 0$.
(1) A positive defined.

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^n} W(x)|u|^p dx,$$

has at least one nontrivial critical point.
(2) A strongly indefined.

$$I(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \frac{1}{p}\int_{\mathbb{R}^n} W(x)|u|^p dx,$$

has at least one nontrivial critical point.

Supplement to the Differences

Case II. Consider $Au + W(x)|u|^{p-2}u = 0$, $W \geq 0$.
(1) A poitive defined.

$$I(x) = \frac{1}{2}\|u\|^2 + \frac{1}{p}\int_{\mathbb{R}^n} W(x)|u|^p dx$$

has only trivial solution .
Proof. If u is a critical point of I, then

$$I(u) - \frac{1}{2}dI(u)\cdot u = (\frac{1}{p} - \frac{1}{2})\int W(x)|u|^p dx \leq 0.$$

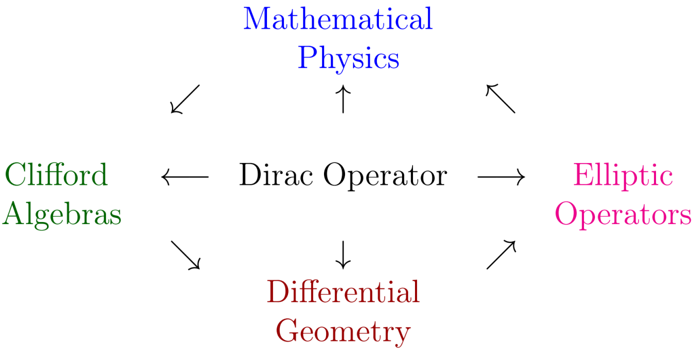
This implies $u = 0$.
(2) A strongly indefined.

$$I(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) + \frac{1}{p}\int_{\mathbb{R}^n} W(x)|u|^p dx,$$

has at least one nontrivial critical point. Only need to consider

$$-I(u) = \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - \frac{1}{p}\int_{\mathbb{R}^n} W(x)|u|^p dx.$$

From Many Aspects



Case I. $V(x) = 0$

Recall $H_\omega = -ic\hbar\alpha \cdot \nabla + mc^2\beta + \omega$ is well-defined on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathscr{D}(H_\omega) = H^1(\mathbb{R}^3, \mathbb{C}^4)$ and formal domain $\mathscr{D}(H_\omega) = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$.

Proposition

$\sigma(H_\omega) = \sigma_e(H_\omega) = \mathbb{R} \setminus (-mc^2 + \omega, mc^2 + \omega)$.

Proof. Symbol of H_0 is denoted by $\hat{H}_0$. By

$$\sigma(H_0) = \overline{\{\lambda \in \mathbb{C} : \exists \xi \in \mathbb{R}^n,\ s.t. \det(\hat{H}_0(\xi) - \lambda I) = 0\}},$$

$$\det(\hat{H}_0(\xi) - \lambda I) = (\lambda^2 - m^2c^4 - |\xi|^2)^2.$$

$\Rightarrow \sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, \infty)$
$\Rightarrow \sigma(H_\omega) = (-\infty, -mc^2 + \omega] \cup [mc^2 + \omega, \infty)$

Case II. Periodic Potential

For $H_\omega = -ic\hbar\alpha \cdot \nabla + mc^2\beta + V(x)\beta + \omega$, we assume

(V_p) $V \in \mathcal{C}^1(\mathbb{R}^3, [0, \infty))$, $V(x)$ is 1-periodic with respect to x_k.

Proposition (Bartsch, Ding, 06, JDE)

$\sigma(H_\omega) = \sigma_c(H_\omega) \subset \mathbb{R} \setminus (-mc^2 + \omega, mc^2 + \omega)$, *and*
$\inf \sigma(H_0) \cap \mathbb{R}^+ \leq mc^2 + \sup_{x \in \mathbb{R}^3} V(x)$.

Case III. Coercive Potential

For $H_\omega = -ic\hbar\alpha \cdot \nabla + mc^2\beta + V(x)\beta + \omega$, we assume

(V_s) $V \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$, for any $b > 0$, meas$(V^b) < \infty$, where $V^b := \{x \in \mathbb{R}^3 : V(x) \leq b\}$.

Proposition (Bartsch, Ding, 06, JDE)

$\sigma(H_\omega) = \sigma_d(H_\omega) = \left\{\omega \pm \mu_n^{1/2} : n \in \mathbb{N}\right\}$, *where* $0 < \mu_1 \leq \cdots \leq \mu_n \to \infty$.

Case IV. Coulomb-type Potential

Set $H_0 = -ic\hbar\alpha\cdot\nabla + mc^2\beta + V(x)$, we assume
(V_b) $\lim_{|x|\to\infty} V(x) = 0$, $-\frac{\nu}{|x|} - K_1 \le V \le K_2 = \sup_{x\in\mathbb{R}^3} V(x)$,
where $K_1, K_2 \ge 0$, $K_1 + K_2 - mc^2 < \sqrt{m^2c^4 - mc^2\nu^2}$,
$\nu \in (0, \sqrt{mc^2})$, $K_1, K_2 \in \mathbb{R}$.

Proposition (Esteban, Lewin, Séré, 21, PLMS)

$\lambda_k(H_0) = \inf_{\substack{Y\subset C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)\\ \dim Y = k}} \sup_{\varphi\in Y\setminus\{0\}} \lambda^T(H_0,\varphi)$, *where*

$$\lambda^T(H_0,\varphi) := \sup_{\substack{\psi=(\varphi,\chi)^T\\ \chi\in C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)}} \frac{(H_0\psi,\psi)}{(\psi,\psi)} \in (K_2 - mc^2, \infty).$$

Case V. Coulomb-type Potential

(V_b') $\lim_{|x|\to\infty} V(x) = 0$, $V \in C(\mathbb{R}^3 \setminus P, \mathbb{R})$, where $P = \{x_i^+\}_{i=1}^I \cup \{x_j^-\}_{j=1}^J$. And

$$\lim_{x\to x_i^+} V(x) = +\infty, \quad \lim_{x\to x_i^+} V(x)|x - x_i^+| \le v_i,$$

$$\lim_{x\to x_j^-} V(x) = -\infty, \quad \lim_{x\to x_j^-} V(x)|x - x_j^-| \le v_j,$$

where $v_i, v_j \in (0, 1)$.

Proposition (Dolbeault, Esteban, Séré, 06, JEMS)

(i) $\sigma_e(A) = (-\infty, -mc^2] \cup [mc^2, \infty)$.
(ii) $\sigma(A) = (-\infty, -mc^2] \cup \{\lambda_k^\pm : k \ge 1\} \cup [mc^2, \infty)$.

Working Space

- **Free Dirac operator** $D = -ic\hbar\alpha \cdot \nabla + mc^2\beta$.
- **The orthogonal decomposition** of $L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$L^2 = L^+ \oplus L^-, \ u = u^+ + u^-,$$

with D is positive (or negative) definite on L^+ (or L^-).
- **Working Space** E is the completion of $\mathscr{D}(|D|^{1/2})$ under the inner product

$$(u, v) := \Re(|D|^{1/2}u, |D|^{1/2}v)_{L^2}.$$

- **The orthogonal decomposition** of $E \cong H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$:

$$E = E^+ \oplus E^-,$$

where $E^{\pm} = E \cap L^{\pm}$.

Working Space

In the Fourier domain $\xi = (\xi_1, \xi_2, \xi_3)$, we have

$$\hat{D}(\xi) = c\hbar\alpha \cdot \xi + mc^2\beta = \begin{pmatrix} mc^2 I_2 & c\hbar\sigma \cdot \xi \\ c\hbar\sigma \cdot \xi & -mc^2 I_2 \end{pmatrix}.$$

The unitary transformation $\mathbf{U}(\xi)$ which diagonalize $\hat{D}(\xi)$ is given explicitly by

$$\mathbf{U}(\xi) = \frac{(mc^2 + \lambda)I_4 + \beta c\alpha \cdot \xi}{\sqrt{2\lambda(mc^2 + \lambda)}} = \Upsilon_+ I_4 + \Upsilon_- \beta \frac{\alpha \cdot \xi}{|\xi|},$$

where $\Upsilon_\pm = \sqrt{\frac{1}{2}(1 \pm mc^2/\lambda)}$. Then we have

$$\mathbf{U}(\xi)\hat{D}(\xi)\mathbf{U}^{-1}(\xi) = \lambda\beta.$$

Working Space

▶ **The orthogonal projections** $P^{\pm}$ on E with kernel $E^{\mp}$ are given by

$$P^{\pm}u(x) = \frac{1}{2}\left(I \pm |D|^{-1}D\right)u(x).$$

$$\widehat{P^{\pm}u}(\xi) = \frac{1}{2}\mathbf{U}^{-1}(\xi)(I_4 \pm \beta)\mathbf{U}(\xi)\hat{u}(\xi).$$

Proposition (Dong, Ding, Guo, 23, JDE)

Let $E_p^{\pm} := E^{\pm} \cap L^p$ *for* $p \in (1, \infty)$. *Then there holds*

$$L^p = cl_p E_p^+ \oplus cl_p E_p^-,$$

where cl_p *denotes the closure with respect to the norm in* L^p. *That is, there exists* $\tau_p > 0$ *for every* $p \in (1, \infty)$ *such that*

$$\tau_p \|u^{\pm}\|_{L^p} \leq \|u\|_{L^p}, \ \forall u \in E \cap L^p.$$

Back to Physics: Lagrangian Formalism

- **Nonlinear Dirac Model** One of its general forms is

$$i c\hbar \gamma^\mu \partial_\mu \psi - mc^2 \psi - G_{\bar\psi} + \partial_\mu (G_{\partial_\mu \bar\psi}) = 0,$$

- **Lagrangian density**

$$\mathcal{L} = i c\hbar \bar\psi \gamma^\mu \partial_\mu \psi - mc^2 \bar\psi \psi - G(\psi, \bar\psi, \partial_\mu \psi, \partial_\mu \bar\psi).$$

- **Solitary wave solutions** $\psi(t,x) = e^{-i\omega t} u(x)$.
- **Stationary nonlinear Dirac equations**

$$-i c\hbar \alpha \cdot \nabla u + mc^2 \beta u + V u - \omega u = F_u(u).$$

- **Dirac operator**

$$H_\omega := -i c\hbar \alpha \cdot \nabla + mc^2 \beta + V - \omega$$

Three Physical Model

- **Dirac-Slater Model** The one-particle Dirac-Slater equation becomes

$$-ic\hbar\alpha\cdot\nabla\psi+mc^2\beta\psi+V_c\psi-C_{ex}|\psi|^{2/3}\psi=\omega\psi,$$

where $C_{ex}=3C_{KS}\left(\frac{3}{4\pi}\right)^{1/3}$, $\psi:\mathbb{R}^3\to\mathbb{C}^4$.

- **Dirac-Soler Model** The one-particle Dirac-Soler equation becomes

$$-ic\hbar\alpha\cdot\nabla\psi+mc^2\beta\psi-g(\overline{\psi}\psi)\gamma^0\psi=\omega\psi,$$

where $\psi:\mathbb{R}^3\to\mathbb{C}^4$.

- **Massive Thirring Model**

$$\left(-i\gamma^5\partial_x+m\gamma^0\right)u+g_c\left(-|u|^2u+\left(u^\dagger\gamma^5u\right)\gamma^5u\right)=\omega u,$$

where $u:\mathbb{R}\to\mathbb{C}^2$.

Nonrelativistic limit

- Physical Meaning.
- Case I: Noncompactness Potentials.
- Case II: Compactness Potentials.
- Case III: Normalized Solutions.
- Applications I: Nonexistence Results.
- Applications II: Nonlinear Schrödinger Equations.

Nonrelativistic limit I (Noncompactness)

Consider the following nonlinear Dirac equation:

$$-ic\alpha\cdot\nabla\psi+mc^2\beta\psi-\omega\psi=|\psi|^{p-2}\psi, \tag{1}$$

where $\psi:\mathbb{R}^3\to\mathbb{C}^4$.
Nonlinear Schrödinger equations:

$$\begin{cases}-\Delta u_1+\nu u_1=2m|u|^{p-2}u_1,\\ -\Delta u_2+\nu u_2=2m|u|^{p-2}u_2,\end{cases} \tag{2}$$

where $u=(u_1,u_2)^T:\mathbb{R}^3\to\mathbb{C}^2$, $\nu>0$ is a constant.

Without Potentials

Theorem (Y. Ding, X. Dong, Q. Guo, CVPDE, 2021)

Let $m, \nu > 0, p \in (2, 5/2]$. *Assume* $\{c_n\}, \{\omega_n\}$ *satisfy*

$$0 < c_n, \omega_n \to +\infty,$$

$$0 < \omega_n < mc_n^2,$$

$$mc_n^2 - \omega_n \to \frac{\nu}{m},$$

where $n \to \infty$. *If* $\{\psi_n = (u_n, v_n)^T\}$ *is a ground state of NDE* (1) *with* ω_n, c_n, *then there is a* m_0, *such that for* $m \leq m_0$,

$$u_n \to u \quad \text{and} \quad v_n \to 0 \quad \text{in} \ H^1(\mathbb{R}^3, \mathbb{C}^2),$$

$n \to \infty$, *where* $u : \mathbb{R}^3 \to \mathbb{C}^2$ *is a wave function of NSE* (2) *with frequency* ν.

Nonrelativistic limit II (Assumptions on Potentials)

(VW_1) $V(x), W(x) > 0$ for all $x \in \mathbb{R}^3$, and $V, W \in L^\infty\left(\mathbb{R}^3, \mathbb{R}\right)$.
(VW_2) If $(A_j) \subset \mathbb{R}^3$ is a sequence of Borel sets such that its Lebesgue measure $|A_j| \leq R$, for all $j \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r \to +\infty} \int_{A_j \cap B_r^c(0)} W(x) = 0, \quad \text{uniformly in } j \in \mathbb{N}.$$

Furthermore, one of the below conditions occurs

(VW_3) $\frac{W}{V} \in L^\infty\left(\mathbb{R}^3, \mathbb{R}\right)$.
(VW_4) There exists $q \in (2, 3)$ such that

$$\frac{W(x)}{V(x)^{3-q}} \to 0 \quad \text{as} \quad |x| \to +\infty.$$

Nonrelativistic limit II (Compactness)

Consider the following nonlinear Dirac equation:

$$-ic\alpha\cdot\nabla\psi + mc^2\beta\psi - \omega\psi + V(x)\psi = W(x)|\psi|^{p-2}\psi, \qquad (3)$$

where $\psi:\mathbb{R}^3\to\mathbb{C}^4$.
A coupled system of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u_1 + 2\nu u_1 + 2mV(x)u_1 = 2mW(x)|u|^{p-2}u_1, \\ -\Delta u_2 + 2\nu u_2 + 2mV(x)u_2 = 2mW(x)|u|^{p-2}u_2, \end{cases} \qquad (4)$$

where $u=(u_1,u_2)^T:\mathbb{R}^3\to\mathbb{C}^2$, $\nu>0$ is a constant.

With Potentials

Theorem (Dong, Ding, Guo, JDE, 2023)

Let $m, \nu > 0, p \in (2, 8/3]$*.Assume* $\{c_n\}, \{\omega_n\}$ *satisfy*

$$0 < c_n, \omega_n \to +\infty,$$

$$0 < \omega_n < mc_n^2,$$

$$mc_n^2 - \omega_n \to \frac{\nu}{m},$$

where $n \to \infty$*. Under the hypothesis* $(VW_1) - (VW_2)$*,* (VW_3) *or* (VW_4)*, and* $\|V\|_\infty < \inf(mc_n^2 - \omega_n)$*, if* $\{\psi_n = (u_n, v_n)^T\}$ *is a sequence of ground states for NDE* (3)*. Then*

$$u_n \to u \quad \text{and} \quad v_n \to 0 \quad \text{in} \quad H^1(\mathbb{R}^3, \mathbb{C}^2),$$

as $n \to \infty$*, where* $u : \mathbb{R}^3 \to \mathbb{C}^2$ *is a solution for the NSE* (4)*.*

Nonrelativistic limit III (Normalized Solutions)

Consider the following constraint Dirac equation

$$\begin{cases} -ic\alpha\cdot\nabla\psi + mc^2\beta\psi - \omega\psi = f(x,|\psi|)\psi, \\ \int_{\mathbb{R}^3} |\psi|^2 dx = 1, \end{cases} \qquad (NDE)_c$$

where $f(x,|\psi|) = \Gamma * (K|\psi|^\kappa)K|\psi|^{\kappa-2} - P|\psi|^{s-2}$.
Nonlinear Schrödinger equations with L^2-constraint:

$$\begin{cases} -\Delta u + \nu u = 2mP|u|^{s-2}u + 2m\Gamma * (K|u|^\kappa)K|u|^{\kappa-2}u, \\ \int_{\mathbb{R}^3} |u|^2 dx = 1, \end{cases} \qquad (NSE)$$

where $u = (u_1, u_2)^T : \mathbb{R}^3 \to \mathbb{C}^2$, $\nu > 0$ is a constant.

Nonrelativistic limit III (Assumptions on Nonlinearities)

Assumptions on nonlinearities

(K_1) $K \in \mathcal{C}^1(\mathbb{R}^3, (0, +\infty))$ and $\lim_{|x|\to\infty} K(x) = 0$.

(P_1) $P \in \mathcal{C}^1(\mathbb{R}^3, (0, +\infty))$ and $\lim_{|x|\to\infty} P(x) = 0$.

(P_2) There exist a constant $C > 0$, a number $\mu \in \left(0, \frac{10-3s}{2}\right)$ such that for small $\varepsilon > 0$, and all $x \in \mathbb{R}^3$, it holds that

$$P(x) \geq C\varepsilon^{\mu} P(\varepsilon x).$$

(Γ_1) $\Gamma \in L_w^{6/(14-6\kappa)}(\mathbb{R}^3) \cap \mathcal{C}(\mathbb{R}^3 \setminus \{0\}, (0, +\infty))$.

Model Nonlinearities

(i) $K(x) = e^{-a|x|}$, where $a > 0$.

(ii) $P(x) = \frac{1}{1+|x|^{\mu}}$, where μ is given in assumption (P_2).

(iii) $\Gamma(x) = \frac{1}{|x|^{\tau}}$, where $\tau \in (0, 7 - 3\kappa)$.

Theorem (Chen, Ding, Guo, Wang, 2023, preprint)

Set $\kappa \in [2, 7/3), s \in (2, 8/3]$. *If assumptions* $(K_1), (P_1), (P_2)$ *and* (Γ_1) *hold, then for a given* $c > 0$ *large enough, there exists* $\omega_c \in (0, mc^2)$ *and a function* $u_c \in H^1(\mathbb{R}^3, \mathbb{C}^4)$*, such that* (ω_c, u_c) *is a normalized solution of* $(NDE)_c$*. In addition, we have*

$$-\infty < \liminf_{c\to\infty}(\omega_c - mc^2) \leqslant \limsup_{c\to\infty}(\omega_c - mc^2) < 0.$$

Theorem (Chen, Ding, Guo, Wang, 2023, preprint)

There is a positive constant $\nu > 0$*, and a sequence* $(\omega_{c_n}, \psi_{c_n} = (u_n, v_n)^T)$ *which is a solution of* $(NDE)_{c_n}$*, such that*

$$u_n \to u, \quad v_n \to 0 \quad \textit{in} \quad H^1(\mathbb{R}^3, \mathbb{C}^2),$$

where $u : \mathbb{R}^3 \to \mathbb{C}^2$ *is a ground state solution of* (NSE)*.*

Applications I: Nonexistence Results.

Questions. What if two components of solitary wave solution equal zero?
Fourier transform+Dirac matrices $\Rightarrow$ only trivial solutions $\Rightarrow$ Nonexistence of solutions of Majorana-type (i.e. $u^+ = u^-$).

$$\widehat{u_1^+}(\xi) = a(\xi)\left(\hat{u}_1 + \frac{\xi_1 - i\xi_2}{b(\xi)}\hat{u}_4 + \frac{\xi_3}{b(\xi)}\hat{u}_3\right), \qquad \widehat{u_1^-}(\xi) = a(\xi)\left(A(\xi)\hat{u}_1 - \frac{\xi_1 - i\xi_2}{b(\xi)}\hat{u}_4 - \frac{\xi_3}{b(\xi)}\hat{u}_3\right),$$

$$\widehat{u_2^+}(\xi) = a(\xi)\left(\hat{u}_2 + \frac{\xi_1 + i\xi_2}{b(\xi)}\hat{u}_3 - \frac{\xi_3}{b(\xi)}\hat{u}_4\right), \qquad \widehat{u_2^-}(\xi) = a(\xi)\left(A(\xi)\hat{u}_2 - \frac{\xi_1 + i\xi_2}{b(\xi)}\hat{u}_3 + \frac{\xi_3}{b(\xi)}\hat{u}_4\right),$$

$$a(\xi) = \frac{1}{2}\left(1 + \frac{mc^2}{\lambda}\right) = \frac{mc^2 + \sqrt{m^2c^4 + c^2|\xi|^2}}{2\sqrt{m^2c^4 + c^2|\xi|^2}},$$

$$A(\xi) = \frac{\lambda - mc^2}{\lambda + mc^2} = \frac{\sqrt{m^2c^4 + c^2|\xi|^2} - mc^2}{mc^2 + \sqrt{m^2c^4 + c^2|\xi|^2}},$$

$$b(\xi) = \frac{\lambda + mc^2}{c} = mc + \sqrt{m^2c^2 + |\xi|^2}.$$

Applications I: Nonexistence Results.

(F) Given $f \in \mathcal{C}\left(H^1(\mathbb{R}^3, \mathbb{C}^4), \mathbb{R}\right)$. For any $r > 0$,

$$f(u) \neq 0, \quad \forall u \in \mathfrak{S}_r^0,$$

where $\mathfrak{S}_r^0 := \left\{u = (u_1, u_2, 0, 0) \in H^1(\mathbb{R}^3, \mathbb{C}^4) : \|u\|_{H^1} = r\right\}.$

Theorem

Let $m, \nu > 0$, $p \in (2, 8/3]$. Assume that $\|V\|_\infty < \nu/(2m)$. Under the hypothesis $(VW_1) - (VW_2)$, (VW_3) or (VW_4) and (F), there exists $c_0 > 2m/\nu$, such that for any

$$c > c_0, \varepsilon \in (0, 1/c_0), \omega \in (mc^2 - \varepsilon - \nu/m, mc^2 + \varepsilon - \nu/m),$$

$(NDE)_c$ possesses no ground state u with $f(u) = 0$.

$$|u_1|^a + |u_2|^b = |u_3|^c + |u_4|^d \quad \text{or} \quad \|u_3\|_{L^2}^e + \|u_4\|_{L^2}^f = \gamma.$$

Applications II: NSE.

Theorem

Set $\kappa \in [2, 7/3), s \in (2, 8/3]$. *If assumptions* $(K_1), (P_1), (P_2)$ *and* (Γ_1) *hold, then there is* $(\nu, u) \in (0, \infty) \times H^1(\mathbb{R}^3, \mathbb{C}^2)$, *solves*

$$\begin{cases} -\Delta u + \nu u = 2mP|u|^{s-2}u + 2m\Gamma * (K|u|^{\kappa})K|u|^{\kappa-2}u, \\ \displaystyle\int_{\mathbb{R}^3} |u|^2 dx = 1, \end{cases}$$

Sketch of the Proof

Key Ingredient 1: Existence of ground state solutions in H^1 for any $m, c > 0$ and $\omega \in (-mc^2, mc^2)$.

$$\inf_{\varphi \in \mathcal{M}} \Phi(\phi) = \inf_{w \in E^+} \sup_{\phi \in E^- \oplus \mathbb{R}_w} \Phi(\phi),$$

$$\mathcal{M} := \left\{ u \in E \backslash E^- : \Phi'(u) \cdot u = 0 \text{ and } \Phi'(u) \cdot \varphi = 0, \ \forall \varphi \in E^- \right\}.$$

Key Ingredient 2: Uniform Boundedness of Solutions.
Step 1. $\{u_n\}$ is bounded in L^p. (Taking test function)
Step 2. $\{u_n\}$ is bounded in L^2. (Variational equality)
Step 3. $\{u_n\}$ is bounded in H^1. $(p \in (2, 8/3])$
Key Ingredient 3: $\|v_n\|_{H^1} = O(\frac{1}{c_n})$, $\inf_n \|u_n\|_{H^1} \geq \rho > 0$.
Key Ingredient 4: New functional Ψ. $\{u_n\}$ is a (PS)-sequence for Ψ + Compactness (Compactness Potential/ Small Mass)

Spectrum Zero Problem

- The *Spectrum Zero Problem*: the existence of nontrivial solutions $u \in X$ to the equation

$$Au = N(u),$$

where X is a Banach space, A is a self-adjoint linear operator and N is a bounded nonlinear operator.

- Two cases: zero belonging to the interior of the essential spectrum and zero belonging to the boundary.
- Simplify the model by setting $c = \hbar = 1$, then the stationary nonlinear Dirac equations becomes

$$-i\alpha \cdot \nabla u + m\beta u - \omega u = F_u(x, u).$$

- The linear operator of this problem $H_\omega = -i\alpha \cdot \nabla + m\beta - \omega$.
- The spectrum of H_ω on L^2:

$$\sigma(H_\omega) = (-\infty, -m - \omega] \cup [m - \omega, \infty).$$

Spectrum Zero Problem of Type I

- Set $F(u) = \hat{F}(u^+, u^-)$, $p \in (2,3)$, and assume that

(F_1) $\hat{F} \in \mathcal{C}^1\left(\mathbb{C}^4 \times \mathbb{C}^4, \mathbb{R}\right)$.

(F_2) There is $a_1, a_2 > 0$, such that for any $s, t \in \mathbb{C}^4$, we have

$$a_1\left(|s|^p + |t|^2\right) \le \hat{F}(s,t) \le a_2(|s|^p + |t|^p + |t|^2).$$

(F_3) There is $b_1, b_2 > 0$, such that for any $s, t \in \mathbb{C}^4$, we have

$$2\hat{F}(s,t) + b_1|s|^p \le \langle \partial_s \hat{F}, s\rangle + \langle \partial_t \hat{F}, t\rangle \le 3\hat{F}(s,t) - b_2\left(|s|^p + |t|^2\right).$$

(F_4) There is $c_1 > 0$, $d_2 \ge d_1 > 0$, such that for any $s, t \in \mathbb{C}^4$, we have

$$\langle \partial_s \hat{F}, s\rangle \le c_1|s|^p + d_1|t|^2, \quad \langle \partial_t \hat{F}, t\rangle \ge d_2\left(|t|^2 - |s|^p\right).$$

Spectrum Zero Problem of Type I

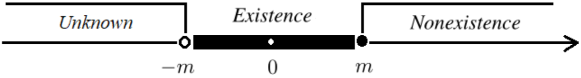
Theorem (Guo, Ke, Ruf, 23, preprint)

Let (F_1)-(F_4) be satisfied, consider the following nonlinear Dirac equation

$$-i\alpha \cdot \nabla u + m\beta u - \omega u = F_u(u).$$

(1) *If $\omega \in [m, \infty)$, then there are only trivial solution $u = 0$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$.*

(2) *If $\omega \in [-m, m)$, then there are at least one nontrivial solution u in $H^1(\mathbb{R}^3, \mathbb{C}^4)$.*



Spectrum Zero Problem of Type I

- **Sketch of the Proof.**

Step 1. Establish Pohozaev's identity and variational identity to solve the case $\omega \geq m$.
Step 2. Set a variational problem when $\omega \in (-m, m)$.
Step 3. Check the topological properties and geometric structure of the functional Φ on E_0.
Step 4. Use the critical point theorem to obtain a $(C)_c$-sequence.
Step 5. Use the Lions's concentration compactness argument to get a new sequence after translation.
Step 6. Show the limit point is the critical point.
Step 7. Peturbation of the functional.
Step 8. Show the uniformly boundedness.
Step 9. Construct a sequence via Step 6.
Step 10. Show the limit point is the critical point when $\omega = -m$.

Spectrum Zero Problem of Type II

- Assumptions on the nonlinearity F:

(F_1) $F \in \mathcal{C}^1\left(\mathbb{R}^3 \times \mathbb{C}^4, \mathbb{R}\right)$ is 1-periodic in $x_i, i = 1, 2, 3$.

(F_2) There are constants $a_1 > 0$ and $2 < \gamma \leq \mu < 3$ such that

$$a_1 |u|^\mu \leq \gamma F(x,u) \leq F_u(x,u) \cdot u, \qquad \text{for all } \ x \in \mathbb{R}^3, u \in \mathbb{R}.$$

(F_3) There are constants $a_2 > 0$ and $2 < p \leq q < 3$ such that

$$|F_u(x,u)| \leq a_2 \left(|u|^{p-1} + |u|^{q-1}\right), \qquad \text{for all } x \in \mathbb{R}^3, u \in \mathbb{R}.$$

Theorem (Dong, Ding, Guo, 24, St. Petersberg Math. J.)

Suppose $(F_1) - (F_3)$ hold. If $\omega = -m$, then nonlinear Dirac equation has a nontrivial (weak) solution $u \in H^1_{loc}\left(\mathbb{R}^3, \mathbb{C}^4\right)$. Moreover, u lies in $L^t\left(\mathbb{R}^3, \mathbb{C}^4\right)$ for $\mu \leq t \leq 3$.

Spectrum Zero Problem

- **New Ingredients:**

Type I.
1. Pohozaev's identity of nonlinear Dirac equations.
2. Critical Point Theorem of strongly indefinite functionals.
3. Perturbation of the functional.

Type II.
1. Choose a proper working space that is neither *too big* nor *too small*.
2. Establish a new embedding theorem for the new working space.
3. Construct a new sequence from the modified functional.

Thanks for your attention !